STA 313F2004 Assignment 3

Do this assignment in preparation for the quiz on Friday, Oct. 22. The hand-written parts are practice for the quiz, and are not to be handed in. The computer part (last question) may be handed in, so bring a printout to the quiz.

1. Let $\mathbf{X}$ and $\mathbf{Y}$ be random matrices of the same dimensions. Show $E(\mathbf{X}+\mathbf{Y}) = E(\mathbf{X}) + E(\mathbf{Y})$.

2. Let $\mathbf{X}$ be a random matrix, and $\mathbf{B}$ be a matrix of constants. Show $E(\mathbf{XB}) = E(\mathbf{X})\mathbf{B}$.

3. If the $p \times 1$ random vector $\mathbf{X}$ has variance-covariance matrix $\mathbf{\Sigma}$ and $\mathbf{A}$ is an $m \times p$ matrix of constants, prove that the variance-covariance matrix of $\mathbf{AX}$ is $\mathbf{A\Sigma A}'$. Start with the definition of a variance-covariance matrix.

4. If the $p \times 1$ random vector $\mathbf{X}$ has mean $\boldsymbol{\mu}$ and variance-covariance matrix $\mathbf{\Sigma}$, show $\mathbf{\Sigma} = E(\mathbf{XX}') - \boldsymbol{\mu}\boldsymbol{\mu}'$.

5. Let $\mathbf{X}$ and $\mathbf{Y}$ be random matrices such that the product $\mathbf{XY}$ can be formed, and let all elements of $\mathbf{X}$ be independent of the elements of $\mathbf{Y}$. Prove $E(\mathbf{XY}) = E(\mathbf{X})E(\mathbf{Y})$.

6. Let $\mathbf{X}$ be a $p \times 1$ random vector with mean $\boldsymbol{\mu}_x$ and variance-covariance matrix $\mathbf{\Sigma}_x$, and let $\mathbf{Y}$ be an $r \times 1$ random vector with mean $\boldsymbol{\mu}_y$ and variance-covariance matrix $\mathbf{\Sigma}_y$. Define $C(\mathbf{X},\mathbf{Y})$ by the $p \times r$ matrix $C(\mathbf{X},\mathbf{Y}) = E\left((\mathbf{X}-\boldsymbol{\mu}_x)(\mathbf{Y}-\boldsymbol{\mu}_y)'\right)$.

   (a) What is the (i,j) element of $C(\mathbf{X},\mathbf{Y})$?

   (b) Find an expression for $V(\mathbf{X}+\mathbf{Y})$ in terms of $\mathbf{\Sigma}_x$, $\mathbf{\Sigma}_y$ and $C(\mathbf{X},\mathbf{Y})$. Show your work.

7. Let X_1 be Normal(μ_1, σ_1^2), and X_2 be Normal(μ_2, σ_2^2), independent of X_1. What is the joint distribution of $Y_1 = X_1 + X_2$ and $Y_2 = X_1 - X_2$? What is required for Y_1 and Y_2 to be independent?

8. Let $\mathbf{X} = (X_1, X_2, X_3)'$ be multivariate normal with

$$\boldsymbol{\mu} = \left[\begin{array}{c} 1 \\ 0 \\ 6 \end{array}\right] \text{ and } \mathbf{\Sigma} = \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{array}\right].$$

   Let $Y_1 = X_1 + X_2$ and $Y_2 = X_2 + X_3$. Find the joint distribution of Y_1 and Y_2.

9. Let $\mathbf{X}_1, \ldots, \mathbf{X}_n$ be independent $MVN(\boldsymbol{\mu}, \mathbf{\Sigma})$ random vectors, and let $\mathbf{\Sigma}$ be fixed and *known*. Derive the maximum likelihood estimate of $\boldsymbol{\mu}$. "Derive" means show all the work. Where do you use the fact that $\mathbf{\Sigma}^{-1}$ is positive definite? Indicate this clearly.

10. Let $\mathbf{Y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\epsilon}$, where $\mathbf{X}$ is an $n \times p$ matrix of known constants, $\boldsymbol{\beta}$ is a $p \times 1$ vector of unknown constants, and $\boldsymbol{\epsilon}$ is multivariate normal with mean zero and covariance matrix $\sigma^2 \mathbf{I}_n$, with $\sigma^2 > 0$ an unknown constant.

    (a) What is the distribution of $\mathbf{Y}$?

    (b) It will be assumed that the rank of $\mathbf{X}$ is $p < n$, so the maximum likelihood estimate (MLE) of $\boldsymbol{\beta}$ is $\hat{\boldsymbol{\beta}} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'Y$. You may use this without proof. Given the MLE of $\boldsymbol{\beta}$, find the MLE of σ^2. Show your work.

    (c) What is the distribution of $\hat{\boldsymbol{\beta}}$? Show the calculations.

    (d) Let $\widehat{\mathbf{Y}} = \mathbf{X}\hat{\boldsymbol{\beta}}$. What is the distribution of $\widehat{\mathbf{Y}}$? Show the calculations.

    (e) Let the vector of residuals $\mathbf{e} = (\mathbf{Y} - \widehat{\mathbf{Y}})$. What is the distribution of $\mathbf{e}$? Show the calculations. Simplify!

11. Let $\mathbf{X}_1, \ldots, \mathbf{X}_n$ be a random sample from a multivariate normal population with mean $\boldsymbol{\mu}$ and variance-covariance matrix $\boldsymbol{\Sigma}$. Using the MLEs

$$\widehat{\boldsymbol{\mu}} = \overline{\mathbf{X}} \text{ and } \widehat{\boldsymbol{\Sigma}} = \frac{1}{n}\sum_{i=1}^{n}(\mathbf{X}_i - \overline{\mathbf{X}})(\mathbf{X}_i - \overline{\mathbf{X}})',$$

    derive the large-sample likelihood ratio test G for testing whether the components of the random vectors $\mathbf{X}_i$ are independent. That is, we want to test whether $\boldsymbol{\Sigma}$ is diagonal. If your simplification of -2 log likelihood does not use the trace of a matrix (see lecture notes) you are leaving something out. What are the degrees of freedom for this test?

12. Write an S function to compute the test you derived in the preceding question. The function should return 3 values: G, the degrees of freedom, and the p-value. Run your function on the sample in `fourvars.dat`; see link to the data on the course web page. Bring a printout showing the definition of your function and illustrating the run on `fourvars.dat`.