STA 312f07 Assignment 6

Do this assignment in preparation for the quiz on Friday, Oct. 26th. The questions are practice for the quiz, and are not to be handed in.

1. Let

$$
\begin{aligned}
Y_1 &= \alpha_1 + \gamma_1 \xi + \zeta_1 \\
Y_2 &= \alpha_2 + \gamma_2 \xi + \zeta_2 \\
X &= \xi + \delta,
\end{aligned}
$$

   where δ, ξ, ζ_1 and ζ_2 are all independent normals, $E(\xi) = \kappa$, $E(\delta) = E(\zeta_1) = E(\zeta_2) = 0$, $Var(\xi) = \phi$, $Var(\zeta_1) = \psi_1$, $Var(\zeta_2) = \psi_2$, $Var(\delta) = \theta_\delta$, and γ_1, γ_2, α_1 and α_2 are fixed constants. The observable variables are X, Y_1 and Y_2.

   (a) Give the mean vector of the observed variables X, Y_1 and Y_2.

   (b) Give the covariance matrix of the observed variables X, Y_1 and Y_2.

   (c) What are the parameters vector of this model? That is, give the parameter vector θ.

   (d) Is this model identified? Answer Yes or No and prove your answer.

2. We can write the log likelihood for the multivariate normal as $\ell(\boldsymbol{\mu}, \boldsymbol{\Sigma})$

$$
\begin{aligned}
&= \log \prod_{i=1}^{n} \frac{1}{|\boldsymbol{\Sigma}|^{\frac{1}{2}} (2\pi)^{\frac{k}{2}}} \exp\left[-\frac{1}{2}(\mathbf{d}_i - \boldsymbol{\mu})' \boldsymbol{\Sigma}^{-1} (\mathbf{d}_i - \boldsymbol{\mu})\right] && (1) \\
&= -\frac{n}{2} \log |\boldsymbol{\Sigma}| - \frac{nk}{2} \log 2\pi - \frac{1}{2} \sum_{i=1}^{n} (\mathbf{d}_i - \boldsymbol{\mu})' \boldsymbol{\Sigma}^{-1} (\mathbf{d}_i - \boldsymbol{\mu}) && (2) \\
&= -\frac{n}{2} \log |\boldsymbol{\Sigma}| - \frac{nk}{2} \log 2\pi - \frac{1}{2} \sum_{i=1}^{n} (\mathbf{d}_i - \overline{\mathbf{d}} + \overline{\mathbf{d}} - \boldsymbol{\mu})' \boldsymbol{\Sigma}^{-1} (\mathbf{d}_i - \overline{\mathbf{d}} + \overline{\mathbf{d}} - \boldsymbol{\mu}) && (3) \\
&= -\frac{n}{2} \log |\boldsymbol{\Sigma}| - \frac{nk}{2} \log 2\pi - \frac{1}{2} \sum_{i=1}^{n} (\mathbf{d}_i - \overline{\mathbf{d}})' \boldsymbol{\Sigma}^{-1} (\mathbf{d}_i - \overline{\mathbf{d}}) \\
&\qquad - \frac{n}{2} (\overline{\mathbf{d}} - \boldsymbol{\mu})' \boldsymbol{\Sigma}^{-1} (\overline{\mathbf{d}} - \boldsymbol{\mu}) && (4) \\
&= -\frac{n}{2} [\log |\boldsymbol{\Sigma}| + k \log 2\pi + tr(\boldsymbol{\Sigma}^{-1} \widehat{\boldsymbol{\Sigma}}) + (\overline{\mathbf{d}} - \boldsymbol{\mu})' \boldsymbol{\Sigma}^{-1} (\overline{\mathbf{d}} - \boldsymbol{\mu})\,]. && (5)
\end{aligned}
$$

   (a) Fill in the work between (1) and (2).

   (b) Fill in the work between (3) and (4).

   (c) Fill in the work between (4) and (5).

   (d) Starting with (5), prove that the MLE of $\boldsymbol{\mu}$ is $\overline{\mathbf{d}}$.

   (e) What is the height of the likelihood function at its highest point?

3. Let

$$\begin{aligned} Y_1 &= \gamma_1 \xi + \zeta_1 \\ Y_2 &= \gamma_2 \xi + \zeta_2 \\ Y_3 &= \gamma_3 \xi + \zeta_3, \end{aligned}$$

where ξ, ζ_1, ζ_3 and ζ_3 are all independent normals with expected value zero, $Var(\xi) = 1$, $Var(\zeta_1) = \psi_1$, $Var(\zeta_2) = \psi_2$, $Var(\zeta_3) = \psi_3$, and γ_1, γ_2 and γ_3 are fixed constants. The observable variables are Y_1, Y_2 and Y_3.

(a) Give the covariance matrix of the observed variables.

(b) What are the parameters vector of this model? That is, give the parameter vector θ.

(c) Is this model identified? Answer Yes or No and prove your answer.