$$y = \alpha + \beta x + \varepsilon$$ for simplicity, assume normality

$$E\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \mu_x \\ \alpha + \beta\mu_x \end{pmatrix}, \quad \operatorname{cov}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \sigma_x^2 & \beta\sigma_x^2 + c \\ & \beta^2\sigma_x^2 + 2\beta c + \sigma_\varepsilon^2 \end{pmatrix} = [v_{ij}]$$

5 eq in 6 unknowns: $\theta = (\alpha, \beta, \mu_x, \sigma_x^2, c, \sigma_\varepsilon^2)$

Focus on

$$\beta v_{11} + c = v_{12} \qquad (1)$$

$$\beta^2 v_{11} + 2\beta c + \sigma_\varepsilon^2 = v_{22} \qquad (1)$$

From (1), c moves as a function of β: $c = v_{12} - \beta v_{11}$

Substitute into (2), getting

$$\beta^2 v_{11} + 2\beta(v_{12} - \beta v_{11}) + \sigma_\varepsilon^2 - v_{22} = 0$$

$$= \beta^2 v_{11} + 2\beta v_{12} - 2\beta^2 v_{11} + \sigma_\varepsilon^2 - v_{22}$$

$$= -v_{11}\beta^2 + 2v_{12}\beta + \sigma_\varepsilon^2 - v_{22}, \text{ so}$$

$$\underbrace{v_{11}}_{A}\beta^2 \underbrace{- 2v_{12}}_{B}\beta + \underbrace{v_{22} - \sigma_\varepsilon^2}_{C} = 0$$

Solutions are $\beta = \dfrac{-B \pm \sqrt{B^2 - 4AC}}{2A}$

$$= \frac{2v_{12} \pm \sqrt{4v_{12}^2 - 4v_{11}(v_{22} - \sigma_\varepsilon^2)}}{2v_{11}} = \frac{v_{12} \pm \sqrt{v_{12}^2 - v_{11}v_{12} + v_{11}\sigma_\varepsilon^2}}{v_{11}}$$

$$= \frac{v_{12} \pm \sqrt{v_{11}\sigma_\varepsilon^2 - |V|}}{v_{11}}$$

For real solutions, need $\sigma_\varepsilon^2 v_{11} \geq |V| \Leftrightarrow \sigma_\varepsilon^2 \geq |V|/v_{11}$

And for any $\sigma_\varepsilon^2 > |V|/v_{11}$, there are 2 β values that θ are compatible with V.

It's clear we can get big + and − values of β by making σ_ε^2 big. Can we also get $\beta = 0$?

- If $v_{12} = 0$, need $v_{11}\sigma_\varepsilon^2 = |V|$
  $\iff \sigma_\varepsilon^2 = |V|/v_{11} > 0$ no problem. The lower bound

- If $v_{12} < 0$, need $v_{12} + \sqrt{v_{11}\sigma_\varepsilon^2 - |V|} = 0$
  $\iff \sqrt{v_{11}\sigma_\varepsilon^2 - |V|} = \underbrace{-v_{12}}_{pos} \iff v_{11}\sigma_\varepsilon^2 - |V| = v_{12}^2$
  $\iff v_{11}\sigma_\varepsilon^2 = v_{12}^2 + |V| \iff \sigma_\varepsilon^2 = \dfrac{v_{12}^2 + |V|}{v_{11}} > 0$ OKAY

- If $v_{12} > 0$, need $v_{12} - \sqrt{v_{11}\sigma_\varepsilon^2 - |V|} = 0$
  $\iff \sqrt{v_{11}\sigma_\varepsilon^2 - |V|} = \underbrace{v_{12}}_{pos} \iff v_{11}\sigma_\varepsilon^2 - |V| = v_{12}^2$
  As before, $\sigma_\varepsilon^2 = \dfrac{v_{12}^2 + |V|}{v_{11}} > 0$ OKAY

For any m & V, so there's a one-d set of points in $\mathbb{R}^6$ connecting β & σ_ε^2, (actually all the points) with β ranging from $-\infty$ to ∞ and σ_ε^2 ranging from $|V|/v_{11}$ to ∞

- μ_x and σ_x^2 remain constant at $\mu_x = m$, and $\sigma_x^2 = v_{11}$
- c and α are linear functions of β
- When $\sigma_\varepsilon^2 = |V|/v_{11}$, $\beta = \dfrac{v_{12}}{v_{11}}$
- When $\sigma_\varepsilon^2 > |V|/v_{11}$, there are 2 β values.

ALL THE PARAMETER VALUES ON THE CURVE PRODUCE m & V. (2 intersecting curves)