Interactions and Factorial ANOVA

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Interactions

- Interaction between explanatory variables means "It depends."
- Relationship between one explanatory variable and the response variable *depends* on the value of the other explanatory variable.
- Can have
  - Quantitative by quantitative
  - Quantitative by categorical
  - Categorical by categorical

Quantitative by Quantitative

$$
\begin{aligned}
Y &= \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_1 x_2 + \epsilon \\
E(Y|\mathbf{x}) &= \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_1 x_2
\end{aligned}
$$

For fixed x_2

$$E(Y|\mathbf{x}) = (\beta_0 + \beta_2 x_2) + (\beta_1 + \beta_3 x_2) x_1$$

Both slope and intercept depend on value of x_2

And for fixed x_1, slope and intercept relating x_2 to E(Y) depend on the value of x_1

Quantitative by Categorical

- One regression line for each category.
- Interaction means slopes are not equal
- Form a product of quantitative variable by each dummy variable for the categorical variable
- For example, three treatments and one covariate: x_1 is the covariate and x_2, x_3 are dummy variables

$$\begin{aligned} Y &= \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 \\ &\quad + \beta_4 x_1 x_2 + \beta_5 x_1 x_3 + \epsilon \end{aligned}$$

General principle

- Interaction between A and B means
  - Relationship of A to Y depends on value of B
  - Relationship of B to Y depends on value of A
- The two statements are formally equivalent

Make a table

$$E(Y|\mathbf{x}) = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 + \beta_4 x_1 x_2 + \beta_5 x_1 x_3$$

Group	x_2	x_3	$E(Y\|\mathbf{x})$
1	1	0	$(\beta_0 + \beta_2) + (\beta_1 + \beta_4)x_1$
2	0	1	$(\beta_0 + \beta_3) + (\beta_1 + \beta_5)x_1$
3	0	0	$\beta_0 \quad + \quad \beta_1 \quad x_1$

Group	x_2	x_3	$E(Y\|\mathbf{x})$
1	1	0	$(\beta_0 + \beta_2) + (\beta_1 + \beta_4)x_1$
2	0	1	$(\beta_0 + \beta_3) + (\beta_1 + \beta_5)x_1$
3	0	0	$\beta_0 + \beta_1 x_1$

What null hypothesis would you test for

- Equal slopes
- Comparing slopes for group one vs three
- Comparing slopes for group one vs two
- Equal regressions
- Interaction between group and x_1

What to do if H_0: $\beta_4=\beta_5=0$ is rejected

- How do you test Group “controlling” for x_1?
- A reasonable choice is to set x_1 to its sample mean, and compare treatments at that point.

Categorical by Categorical

- Naturally part of factorial ANOVA in experimental studies
- Also applies to purely observational data

Factorial ANOVA

More than one categorical explanatory variable

Factorial ANOVA

- Categorical explanatory variables are called **factors**
- More than one at a time
- Primarily for true experiments, but also used with observational data

- If there are observations at all combinations of explanatory variable values, it's called a *complete* factorial design (as opposed to a fractional factorial).

The potato study

- Cases are potatoes
- Inoculate with bacteria, store for a fixed time period.
- Response variable is percent surface area with visible rot.
- Two explanatory variables, randomly assigned
  - Bacteria Type (1, 2, 3)
  - Temperature (1=Cool, 2=Warm)

Two-factor design

	Bacteria Type		
Temp	1	2	3
1=Cool			
2=Warm			

Six treatment conditions

Factorial experiments

- Allow more than one factor to be investigated in the same study: Efficiency!
- Allow the scientist to see whether the effect of an explanatory variable *depends* on the value of another explanatory variable: Interactions
- Thank you again, Mr. Fisher.

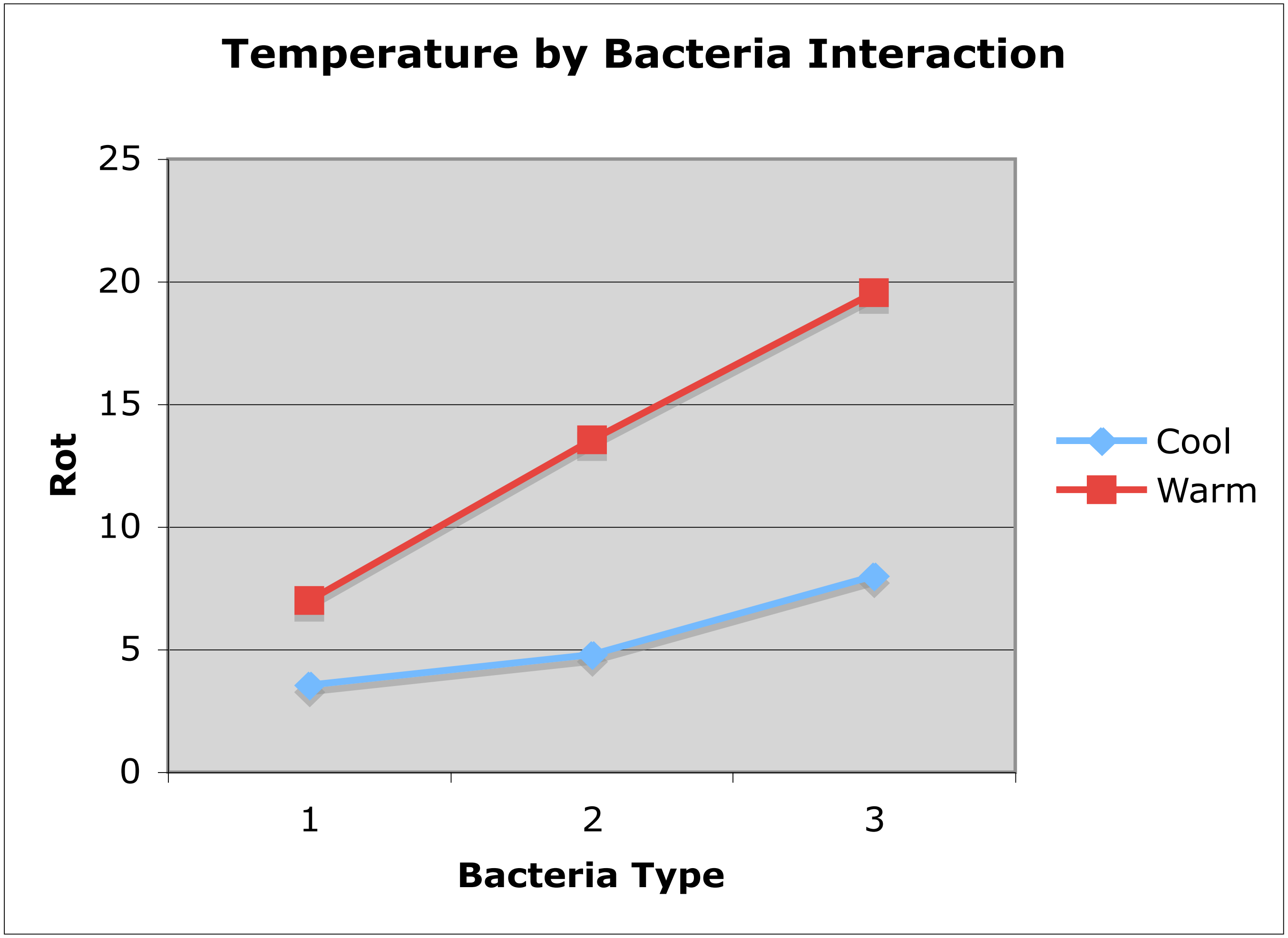
Normal with equal variance and conditional (cell) means $\mu_{i,j}$

	Bacteria Type			
Temp	1	2	3	
1=Cool	$\mu_{1,1}$	$\mu_{1,2}$	$\mu_{1,3}$	$\frac{\mu_{1,1}+\mu_{1,2}+\mu_{1,3}}{3}$
2=Warm	$\mu_{2,1}$	$\mu_{2,2}$	$\mu_{2,3}$	$\frac{\mu_{2,1}+\mu_{2,2}+\mu_{2,3}}{3}$
	$\frac{\mu_{1,1}+\mu_{2,1}}{2}$	$\frac{\mu_{1,2}+\mu_{2,2}}{2}$	$\frac{\mu_{1,3}+\mu_{2,3}}{2}$	μ

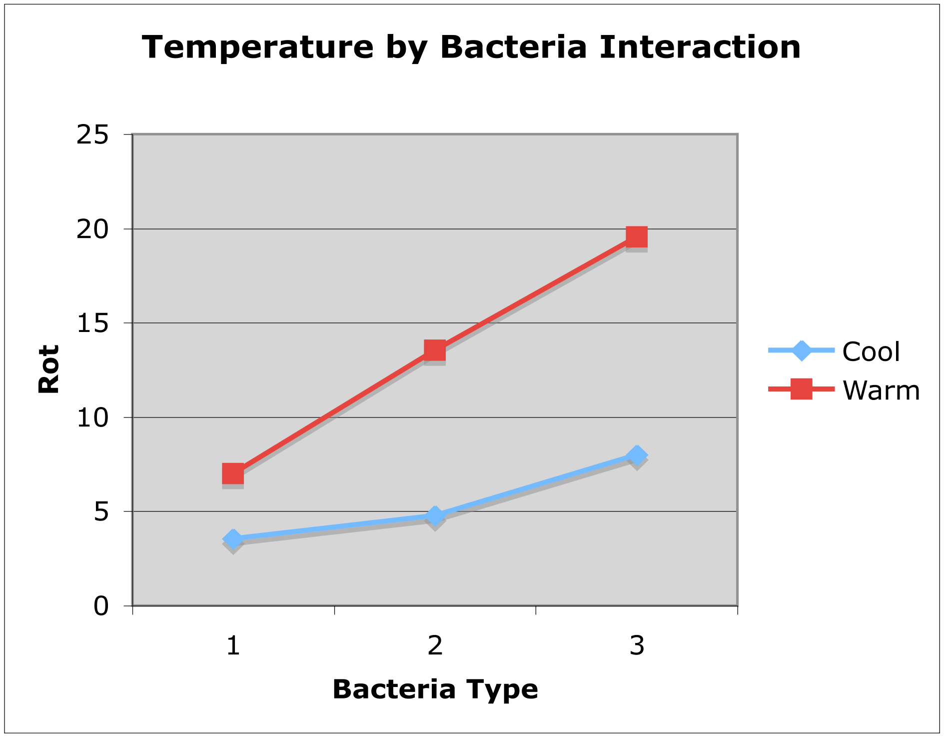
Tests

- Main effects: Differences among marginal means
- Interactions: Differences between differences (What is the effect of Factor A? **It depends** on the level of Factor B.)

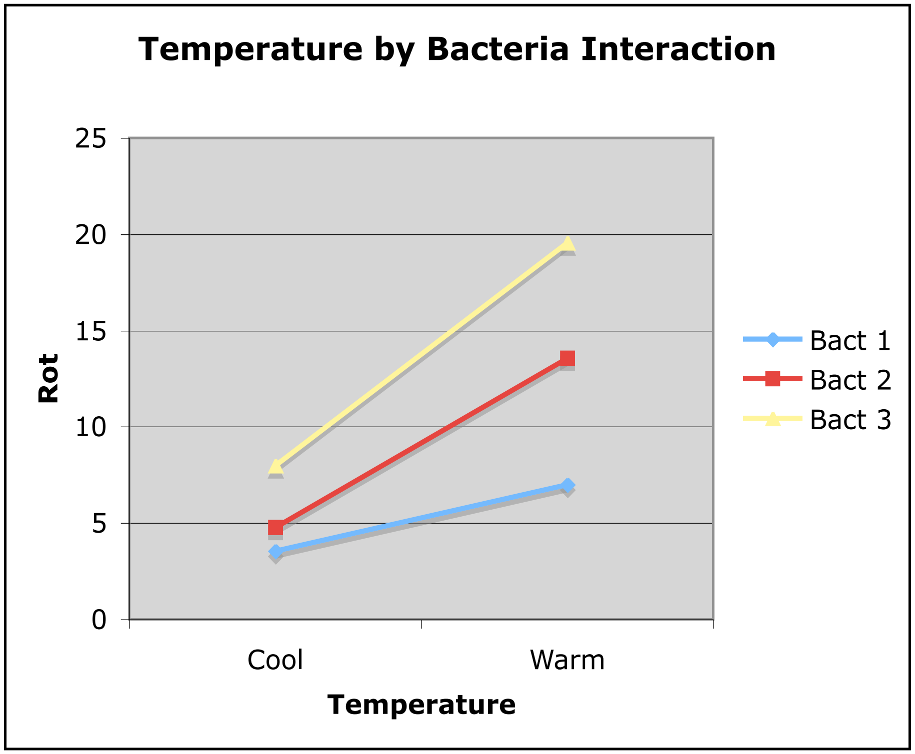
To understand the interaction, plot the means



Either Way

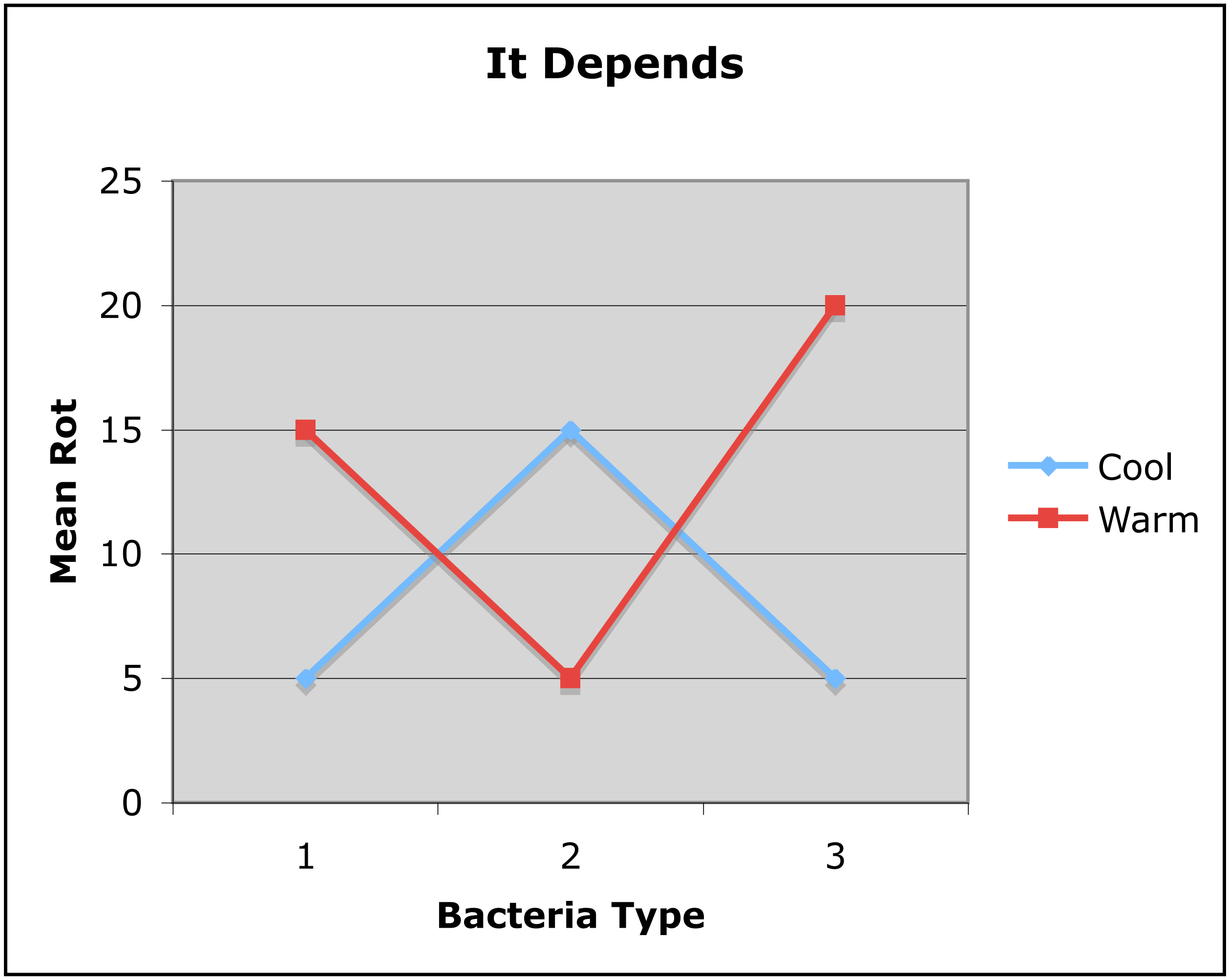


Temperature by Bacteria Interaction
Rot
25
20
15
10
5
0
1
2
3
Bacteria Type
Cool
Warm

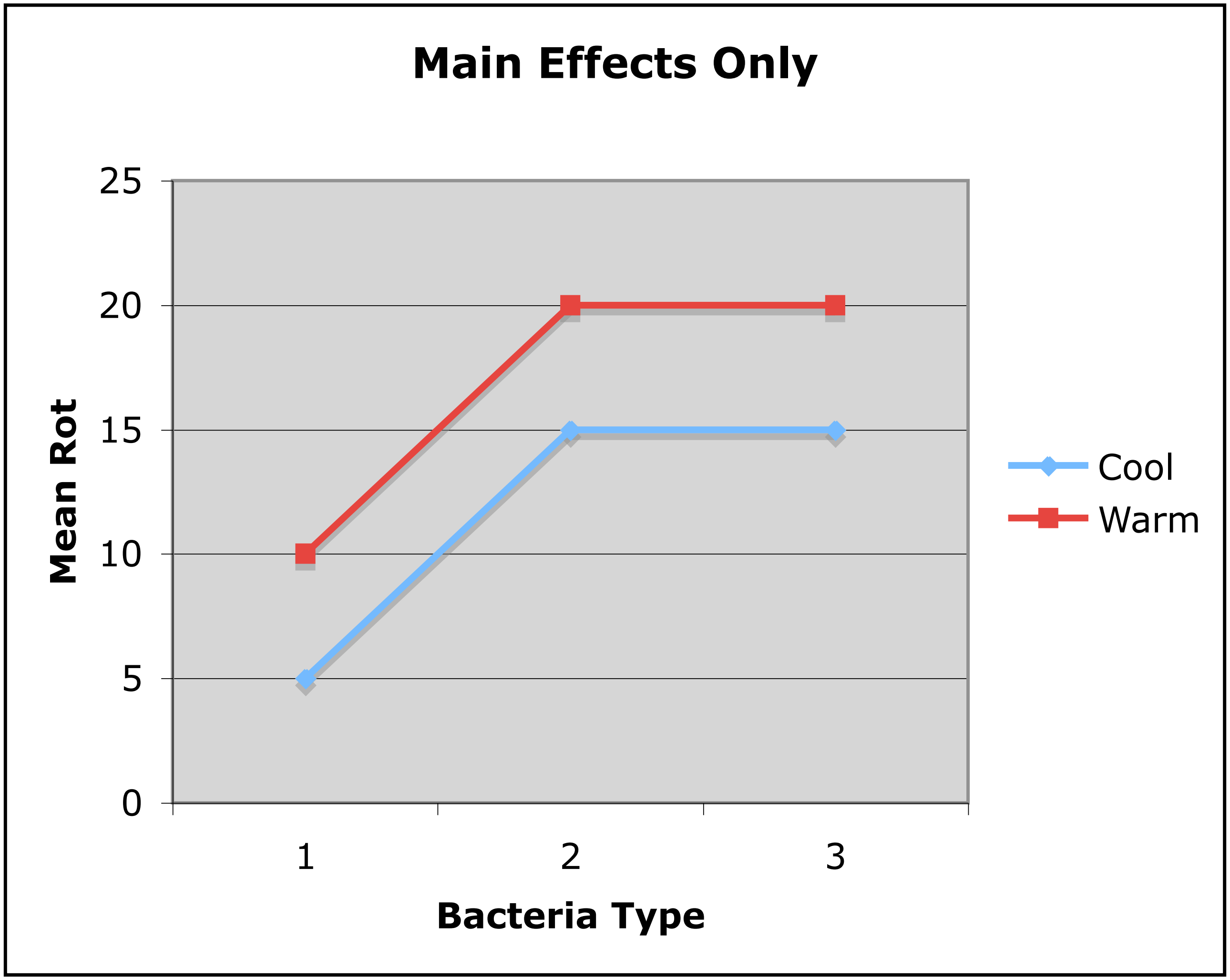


Temperature by Bacteria Interaction
Rot
25
20
15
10
5
0
Cool
Warm
Temperature
Bact 1
Bact 2
Bact 3

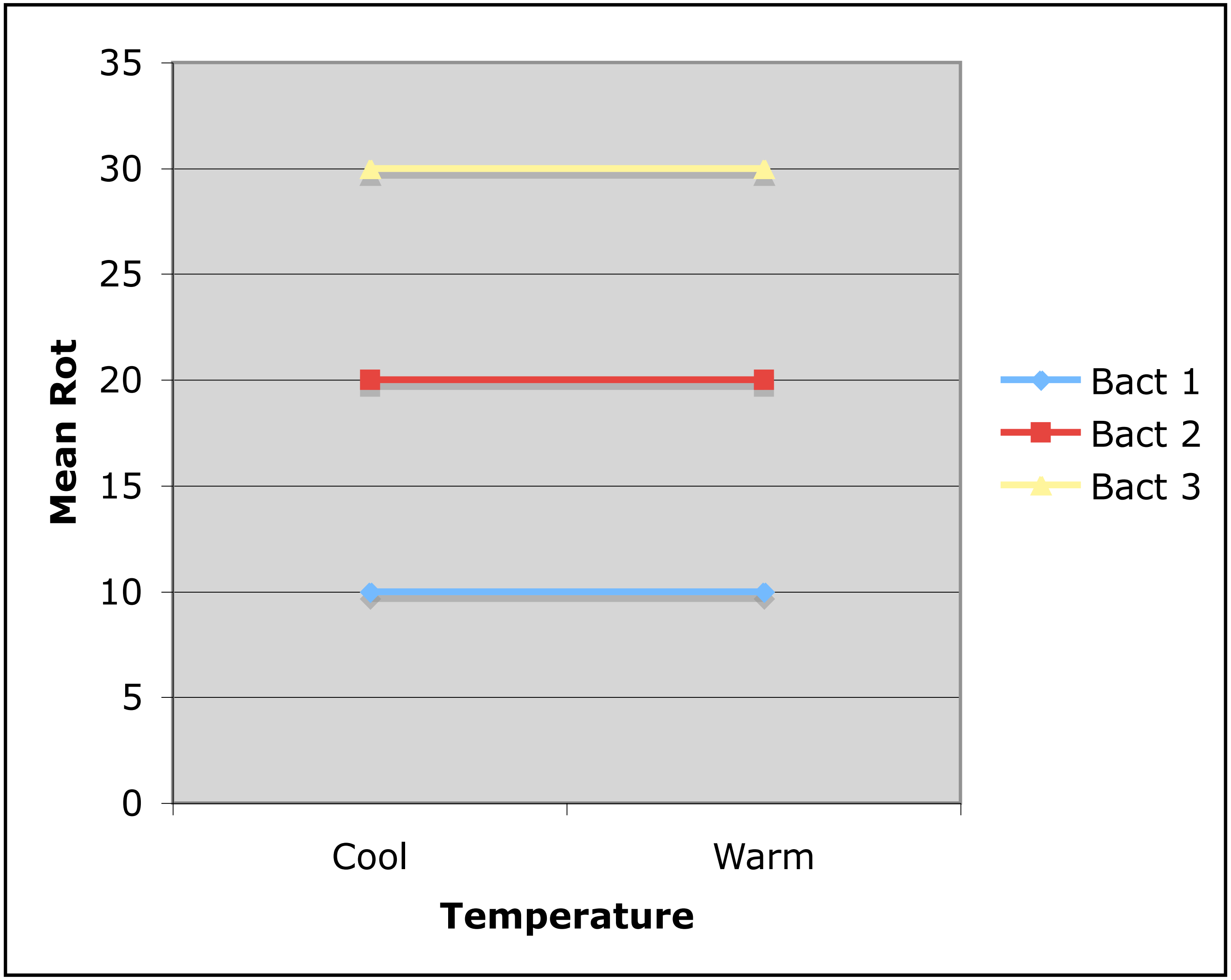
Non-parallel profiles = Interaction



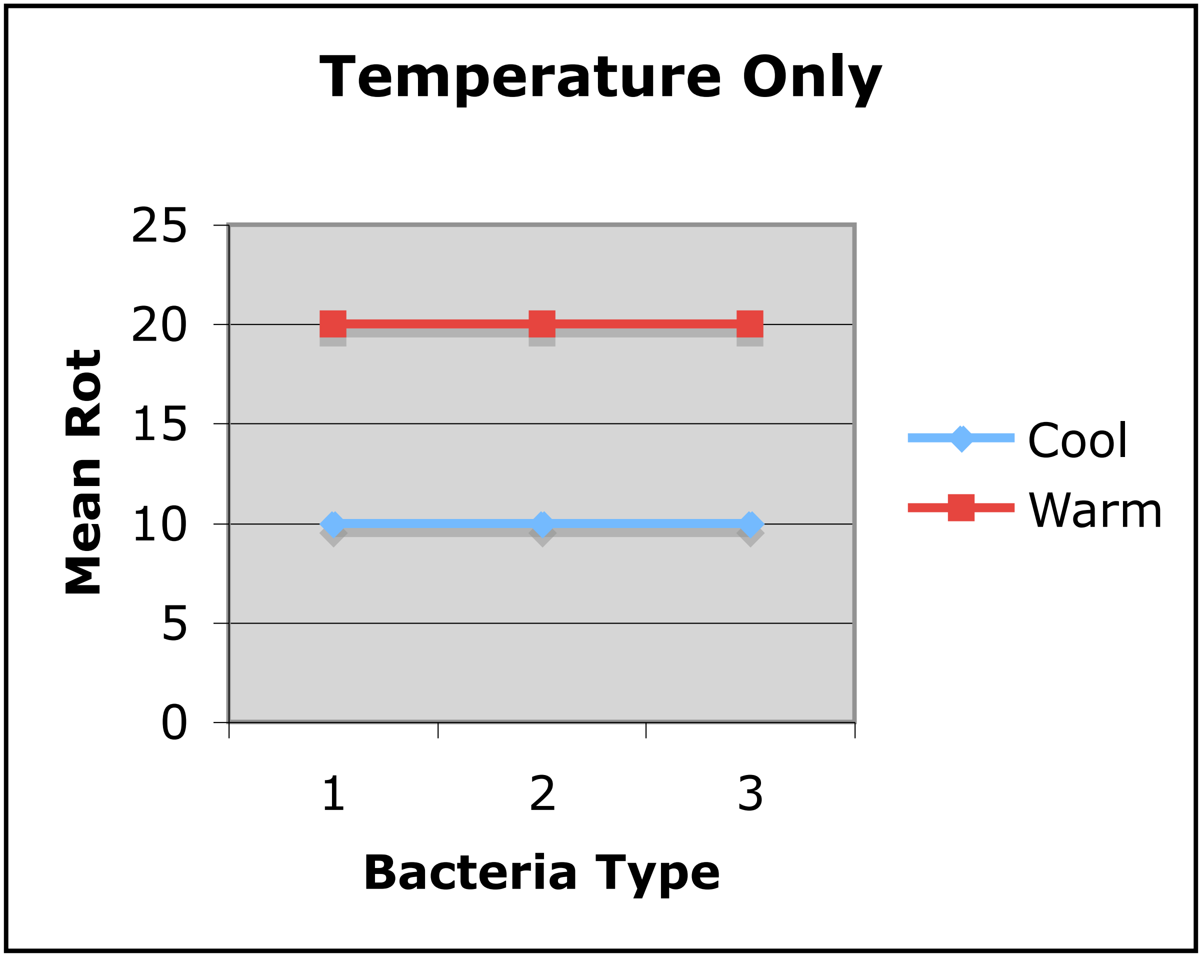
Main effects for both variables, no interaction



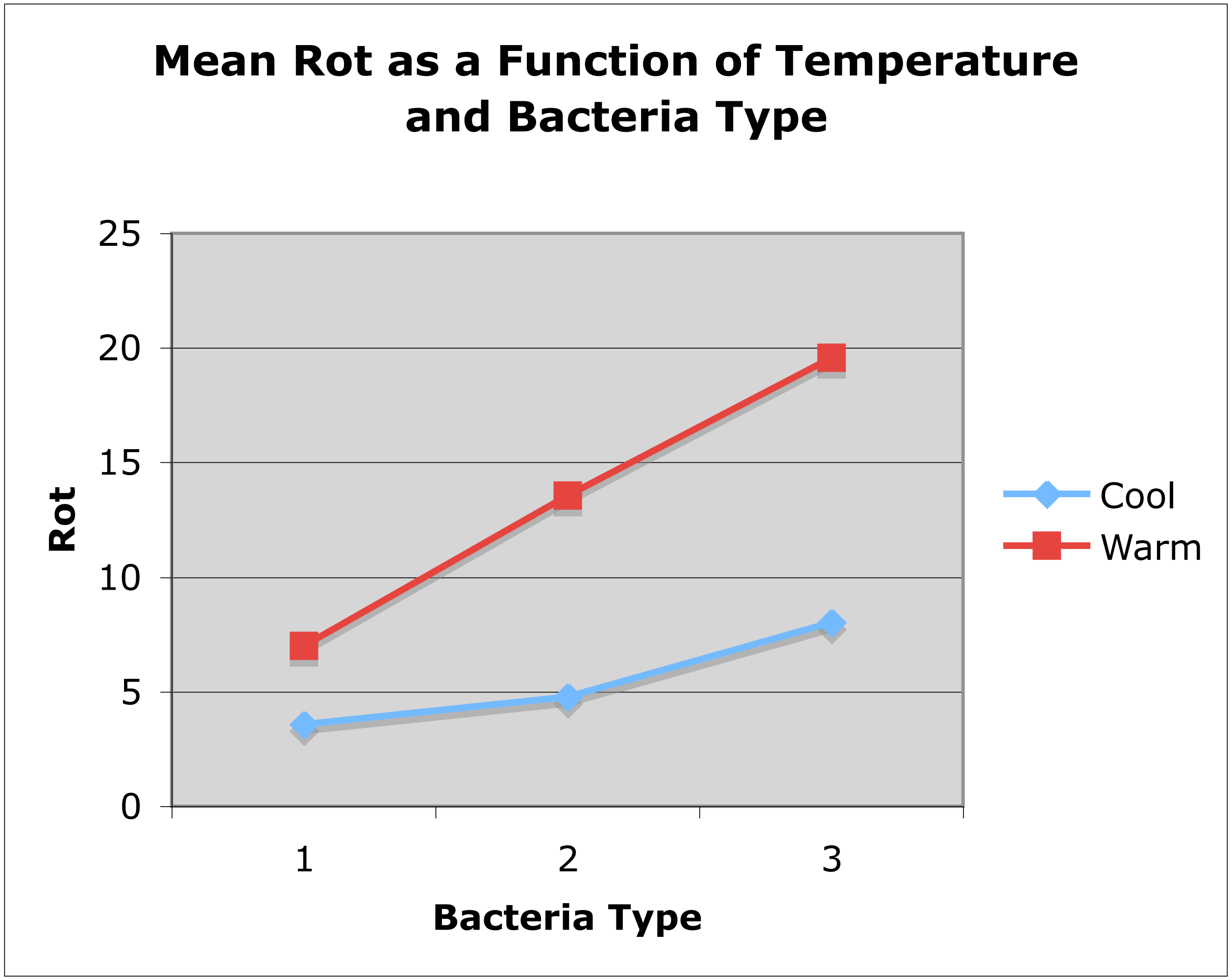
Main effect for Bacteria only



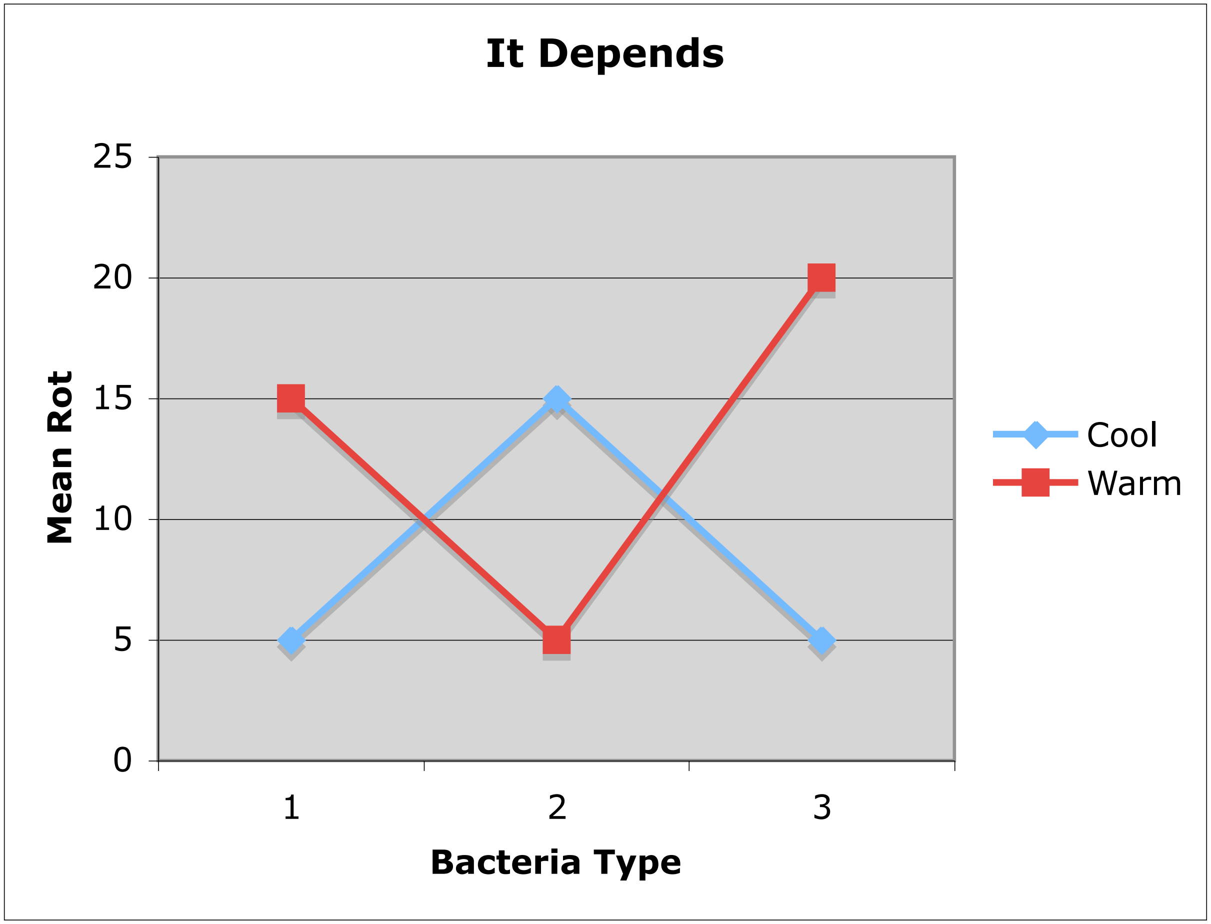
Main Effect for Temperature Only



Both Main Effects, and the Interaction



Should you interpret the main effects?



Contrasts

$$c = a_1\mu_1 + a_2\mu_2 + \cdots + a_p\mu_p$$

$$\widehat{c} = a_1\overline{Y}_1 + a_2\overline{Y}_2 + \cdots + a_p\overline{Y}_p$$

$$\text{where } a_1 + a_2 + \cdots + a_p = 0$$

In a one-factor design

- Mostly, what you want are tests of contrasts,
- Or collections of contrasts.
- You could do it with any dummy variable coding scheme.
- Cell means coding is often most convenient.
- With $\boldsymbol{\beta}=\boldsymbol{\mu}$, test H_0: $\mathbf{L}\boldsymbol{\beta}=\mathbf{h}$

- Can get a confidence interval for any single contrast using the *t* distribution.

Testing Contrasts in Factorial Designs

	Bacteria Type			
Temp	1	2	3	
1=Cool	$\mu_{1,1}$	$\mu_{1,2}$	$\mu_{1,3}$	$\frac{\mu_{1,1}+\mu_{1,2}+\mu_{1,3}}{3}$
2=Warm	$\mu_{2,1}$	$\mu_{2,2}$	$\mu_{2,3}$	$\frac{\mu_{2,1}+\mu_{2,2}+\mu_{2,3}}{3}$
	$\frac{\mu_{1,1}+\mu_{2,1}}{2}$	$\frac{\mu_{1,2}+\mu_{2,2}}{2}$	$\frac{\mu_{1,3}+\mu_{2,3}}{2}$	μ

- Differences between marginal means are definitely contrasts
- Interactions are also sets of contrasts

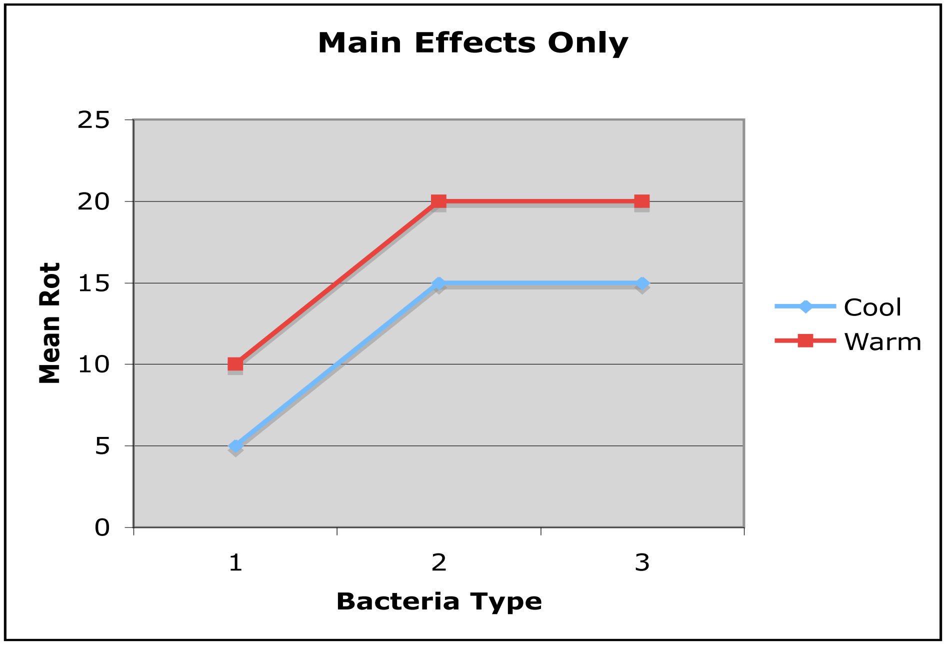
Interactions are sets of Contrasts

	Bacteria Type			
Temp	1	2	3	
1=Cool	$\mu_{1,1}$	$\mu_{1,2}$	$\mu_{1,3}$	$\frac{\mu_{1,1}+\mu_{1,2}+\mu_{1,3}}{3}$
2=Warm	$\mu_{2,1}$	$\mu_{2,2}$	$\mu_{2,3}$	$\frac{\mu_{2,1}+\mu_{2,2}+\mu_{2,3}}{3}$
	$\frac{\mu_{1,1}+\mu_{2,1}}{2}$	$\frac{\mu_{1,2}+\mu_{2,2}}{2}$	$\frac{\mu_{1,3}+\mu_{2,3}}{2}$	μ

- $H_0 : \mu_{1,1} - \mu_{2,1} = \mu_{1,2} - \mu_{2,2} = \mu_{1,3} - \mu_{2,3}$

- $H_0 : \mu_{1,2} - \mu_{1,1} = \mu_{2,2} - \mu_{2,1}$ and
  $\mu_{1,3} - \mu_{1,2} = \mu_{2,3} - \mu_{2,2}$

Interactions are sets of Contrasts



- $H_0 : \mu_{1,1} - \mu_{2,1} = \mu_{1,2} - \mu_{2,2} = \mu_{1,3} - \mu_{2,3}$

- $H_0 : \mu_{1,2} - \mu_{1,1} = \mu_{2,2} - \mu_{2,1}$ and
  $\mu_{1,3} - \mu_{1,2} = \mu_{2,3} - \mu_{2,2}$

Equivalent statements

- The effect of A depends upon B
- The effect of B depends on A

$$H_0 : \mu_{1,1} - \mu_{2,1} = \mu_{1,2} - \mu_{2,2} = \mu_{1,3} - \mu_{2,3}$$

$$H_0 : \mu_{1,2} - \mu_{1,1} = \mu_{2,2} - \mu_{2,1} \text{ and}$$
$$\mu_{1,3} - \mu_{1,2} = \mu_{2,3} - \mu_{2,2}$$

Three factors: A, B and C

- There are three (sets of) main effects: One each for A, B, C
- There are three two-factor interactions
  - A by B (Averaging over C)
  - A by C (Averaging over B)
  - B by C (Averaging over A)
- There is one three-factor interaction: AxBxC

Meaning of the 3-factor interaction

- The form of the A x B interaction depends on the value of C
- The form of the A x C interaction depends on the value of B
- The form of the B x C interaction depends on the value of A
- These statements are equivalent. Use the one that is easiest to understand.

To graph a three-factor interaction

- Make a separate mean plot (showing a 2-factor interaction) for each value of the third variable.
- In the potato study, a graph for each type of potato

Four-factor design

- Four sets of main effects
- Six two-factor interactions
- Four three-factor interactions
- One four-factor interaction: The nature of the three-factor interaction depends on the value of the 4th factor
- There is an F test for each one
- And so on …

As the number of factors increases

- The higher-way interactions get harder and harder to understand
- All the tests are still tests of sets of contrasts (differences between differences of differences …)
- But it gets harder and harder to write down the contrasts
- Effect coding becomes easier

Effect coding

Like indicator dummy variables with intercept, but put -1 for the last category.

Bact	B_1	B_2
1	1	0
2	0	1
3	-1	-1

Temperature	T
1=Cool	1
2=Warm	-1

$$E(Y|\mathbf{X}=\mathbf{x}) = \beta_0 + \beta_1 B_1 + \beta_2 B_2 + \beta_3 T + \beta_4 B_1 T + \beta_5 B_2 T$$

Interaction effects are products of dummy variables

$$E(Y|\mathbf{X}=\mathbf{x}) = \beta_0 + \beta_1 B_1 + \beta_2 B_2 + \beta_3 T + \beta_4 B_1 T + \beta_5 B_2 T$$

- The A x B interaction: Multiply each dummy variable for A by each dummy variable for B
- Use these products as additional explanatory variables in the multiple regression
- The A x B x C interaction: Multiply each dummy variable for C by each product term from the A x B interaction
- Test the sets of product terms simultaneously

Make a table

$$E(Y|\mathbf{X}=\mathbf{x}) = \beta_0 + \beta_1 B_1 + \beta_2 B_2 + \beta_3 T + \beta_4 B_1 T + \beta_5 B_2 T$$

Bact	Temp	B_1	B_2	T	B_1T	B_2T	$E(Y\|\mathbf{X}=\mathbf{x})$
1	1	1	0	1	1	0	$\beta_0 + \beta_1 + \beta_3 + \beta_4$
1	2	1	0	-1	-1	0	$\beta_0 + \beta_1 - \beta_3 - \beta_4$
2	1	0	1	1	0	1	$\beta_0 + \beta_2 + \beta_3 + \beta_5$
2	2	0	1	-1	0	-1	$\beta_0 + \beta_2 - \beta_3 - \beta_5$
3	1	-1	-1	1	-1	-1	$\beta_0 - \beta_1 - \beta_2 + \beta_3 - \beta_4 - \beta_5$
3	2	-1	-1	-1	1	1	$\beta_0 - \beta_1 - \beta_2 - \beta_3 + \beta_4 + \beta_5$

Cell and Marginal Means

	Bacteria Type			
Tmp	1	2	3	
1=C	$\beta_0+\beta_1+\beta_3+\beta_4$	$\beta_0+\beta_2+\beta_3+\beta_5$	$\beta_0-\beta_1-\beta_2$ $+\beta_3-\beta_4-\beta_5$	β_0 $+\beta_3$
2=W	$\beta_0+\beta_1-\beta_3-\beta_4$	$\beta_0+\beta_2-\beta_3-\beta_5$	$\beta_0-\beta_1-\beta_2$ $-\beta_3+\beta_4+\beta_5$	β_0 $-\beta_3$
	$\beta_0+\beta_1$	$\beta_0+\beta_2$	$\beta_0-\beta_1-\beta_2$	β_0

We see

- Intercept is the grand mean
- Regression coefficients for the dummy variables are deviations of the marginal means from the grand mean
- What about the interactions?

$$E(Y|\mathbf{X}=\mathbf{x}) = \beta_0 + \beta_1 B_1 + \beta_2 B_2 + \beta_3 T + \beta_4 B_1 T + \beta_5 B_2 T$$

A bit of algebra shows

$\mu_{1,1} - \mu_{2,1} = \mu_{1,2} - \mu_{2,2}$ is equivalent to $\beta_4 = \beta_5$

$\mu_{1,2} - \mu_{2,2} = \mu_{1,3} - \mu_{2,3}$ is equivalent to $\beta_4 = -\beta_5$

So $\beta_4 = \beta_5 = 0$

Factorial ANOVA with effect coding is pretty automatic

- You don’t have to make a table unless asked.
- It always works as you expect it will.
- Hypothesis tests are the same as testing sets of contrasts.
- Covariates present no problem. Main effects and interactions have their usual meanings, “controlling” for the covariates.
- Plot the “least squares means” (Y-hat at x-bar values for covariates).

Again

- Intercept is the grand mean
- Regression coefficients for the dummy variables are deviations of the marginal means from the grand mean
- Test of main effect(s) is test of the dummy variables for a factor.
- Interaction effects are products of dummy variables.

Balanced vs. Unbalanced Experimental Designs

- Balanced design: Cell sample sizes are proportional (maybe equal)
- Explanatory variables have zero relationship to one another
- Numerator SS in ANOVA are independent
- Everything is nice and simple
- Most experimental studies are designed this way.
- As soon as somebody drops a test tube, it’s no longer true

Analysis of unbalanced data

- When explanatory variables are related, there is potential ambiguity.
- A is related to Y, B is related to Y, and A is related to B.
- Who gets credit for the portion of variation in Y that could be explained by either A or B?
- With a regression approach, whether you use contrasts or dummy variables (equivalent), the answer is **nobody**.
- Think of full, reduced models.
- Equivalently, general linear test

Some software is designed for balanced data

- The special purpose formulas are much simpler.
- They were very useful *in the past*.
- Since most data are at least a little unbalanced, thy are a recipe for trouble.
- Most textbook data are balanced, so they cannot tell you what your software is really doing.
- R's `anova` and `aov` functions are designed for balanced data, though `anova` applied to `lm` objects can give you what you want if you use it with care.

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