STA 2101/442 Assignment 5[1]

The questions on this assignment are practice for the quiz on Friday October 13th, and are not to be handed in. Please do the problems using the formula sheet as necessary. A copy of the formula sheet will be distributed with the quiz. As usual, you may use anything on the formula sheet unless you are directly asked to prove it.

1. Suppose $X_1, \ldots, X_n$ are a random sample from a distribution with mean μ and variance σ^2. The central limit theorem says $\sqrt{n}\left(\overline{X}_n - \mu\right) \stackrel{d}{\rightarrow} T \sim N(0, \sigma^2)$. One version of the delta method says that if $g(x)$ is a function whose derivative is continuous in a neighbourhood of $x = \mu$, then $\sqrt{n}\left(g(\overline{X}_n) - g(\mu)\right) \stackrel{d}{\rightarrow} g'(\mu)T$. In many applications, both μ and σ^2 are functions of some parameter θ.

   (a) Let $X_1, \ldots, X_n$ be a random sample from a Bernoulli distribution with parameter θ. Find the limiting distribution of

   $$Z_n = 2\sqrt{n}\left(\sin^{-1}\sqrt{\overline{X}_n} - \sin^{-1}\sqrt{\theta}\right).$$

   Hint: $\frac{d}{dx}\sin^{-1}(x) = \frac{1}{\sqrt{1-x^2}}$. The measurements are in radians, not degrees.

   (b) In a coffee taste test, 100 coffee drinkers tasted coffee made with two different blends of coffee beans, the old standard blend and a new blend. We will adopt a Bernoulli model for these data, with θ denoting the probability that a customer will prefer the new blend. Suppose 60 out of 100 consumers preferred the new blend of coffee beans. Using your answer to the first part of this question, test $H_0 : \theta = \frac{1}{2}$ using a variance-stabilized test statistic. Give the value of the test statistic (a number), and state whether you reject H_0 at the usual $\alpha = 0.05$ significance level. In plain, non-statistical language, what do you conclude? This is a statement about preference for types of coffee, and of course you will draw a directional conclusion if possible.

   (c) If the probability of an event is p, the *odds* of the event is (are?) defined as $p/(1-p)$. Suppose again that $X_1, \ldots, X_n$ are a random sample from a Bernoulli distribution with parameter θ. In this case the *log odds* of $X_i = 1$ would be estimated by

   $$Y_n = \log \frac{\overline{X}_n}{1 - \overline{X}_n}.$$

   That's the natural log, of course. Find the approximate large-sample distribution (that is, the asymptotic distribution) of Y_n. It's normal, of course. Your job is to give the approximate (that is, asymptotic) mean and variance of Y_n.

   (d) Again using the Taste Test data, give a 95% confidence interval for the log odds of preferring the new brand. Your answer is a pair of numbers.

(e) Let $X_1, \ldots, X_n$ be a random sample from an exponential distribution with parameter θ, so that $E(X_i) = \theta$ and $Var(X_i) = \theta^2$.

i. Find a variance-stabilizing transformation. That is, find a function $g(x)$ such that the limiting distribution of

$$Y_n = \sqrt{n}\left(g(\overline{X}_n) - g(\theta)\right)$$

does not depend on θ.

ii. According to a Poisson process model for calls answered by a service technician, service times (that is, time intervals between taking 2 successive calls; there is always somebody on hold) are independent exponential random variables with mean θ. In 50 successive calls, one technician's mean service time was 3.4 minutes. Test whether this technician's mean service time differs from the mandated average time of 3 minutes. Use your answer to the first part of this question.

2. If the $p \times 1$ random vector $\mathbf{x}$ has variance-covariance matrix $\boldsymbol{\Sigma}$ and $\mathbf{A}$ is an $m \times p$ matrix of constants, prove that the variance-covariance matrix of $\mathbf{Ax}$ is $\mathbf{A\Sigma A}^\top$. Start with the definition of a variance-covariance matrix:

$$cov(\mathbf{Z}) = E(\mathbf{Z} - \boldsymbol{\mu}_z)(\mathbf{Z} - \boldsymbol{\mu}_z)^\top.$$

3. If the $p \times 1$ random vector $\mathbf{x}$ has mean $\boldsymbol{\mu}$ and variance-covariance matrix $\boldsymbol{\Sigma}$, show $\boldsymbol{\Sigma} = E(\mathbf{x}\mathbf{x}^\top) - \boldsymbol{\mu}\boldsymbol{\mu}^\top$.

4. Let the $p \times 1$ random vector $\mathbf{x}$ have mean $\boldsymbol{\mu}$ and variance-covariance matrix $\boldsymbol{\Sigma}$, and let $\mathbf{c}$ be a $p \times 1$ vector of constants. Find $cov(\mathbf{x} + \mathbf{c})$. Show your work.

5. Let the $p \times 1$ random vector $\mathbf{x}$ have mean $\boldsymbol{\mu}$ and variance-covariance matrix $\boldsymbol{\Sigma}$; let $\mathbf{A}$ be a $q \times p$ matrix of constants and let $\mathbf{B}$ be an $r \times p$ matrix of constants. Derive a nice simple formula for $cov(\mathbf{Ax}, \mathbf{Bx})$.

6. Let $\mathbf{x}$ be a $p \times 1$ random vector with mean $\boldsymbol{\mu}_x$ and variance-covariance matrix $\boldsymbol{\Sigma}_x$, and let $\mathbf{y}$ be a $q \times 1$ random vector with mean $\boldsymbol{\mu}_y$ and variance-covariance matrix $\boldsymbol{\Sigma}_y$. Let $\boldsymbol{\Sigma}_{xy}$ denote the $p \times q$ matrix $cov(\mathbf{x}, \mathbf{y}) = E\left((\mathbf{x} - \boldsymbol{\mu}_x)(\mathbf{y} - \boldsymbol{\mu}_y)^\top\right)$.

(a) What is the (i, j) element of $\boldsymbol{\Sigma}_{xy}$? You don't need to show any work; just write down the answer.

(b) Find an expression for $cov(\mathbf{x} + \mathbf{y})$ in terms of $\boldsymbol{\Sigma}_x$, $\boldsymbol{\Sigma}_y$ and $\boldsymbol{\Sigma}_{xy}$. Show your work.

(c) Simplify further for the special case where $Cov(X_i, Y_j) = 0$ for all i and j.

(d) Let $\mathbf{c}$ be a $p \times 1$ vector of constants and $\mathbf{d}$ be a $q \times 1$ vector of constants. Find $cov(\mathbf{x} + \mathbf{c}, \mathbf{y} + \mathbf{d})$. Show your work.

7. Let $\mathbf{x} = (X_1, X_2, X_3)^\top$ be multivariate normal with

$$\boldsymbol{\mu} = \begin{pmatrix} 1 \\ 0 \\ 6 \end{pmatrix} \text{ and } \boldsymbol{\Sigma} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Let $Y_1 = X_1 + X_2$ and $Y_2 = X_2 + X_3$. Find the joint distribution of Y_1 and Y_2.

8. Let X_1 be Normal(μ_1, σ_1^2), and X_2 be Normal(μ_2, σ_2^2), independent of X_1. What is the joint distribution of $Y_1 = X_1 + X_2$ and $Y_2 = X_1 - X_2$? What is required for Y_1 and Y_2 to be independent? Hint: Use matrices.

9. Show that if $\mathbf{w} \sim N_p(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ with $\boldsymbol{\Sigma}$ positive definite, $Y = (\mathbf{w} - \boldsymbol{\mu})^\top \boldsymbol{\Sigma}^{-1}(\mathbf{w} - \boldsymbol{\mu})$ has a chi-squared distribution with p degrees of freedom.

10. You know that if $\mathbf{w} \sim N_p(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, then $\mathbf{A}\mathbf{w} + \mathbf{c} \sim N_r(\mathbf{A}\boldsymbol{\mu} + \mathbf{c}, \mathbf{A}\boldsymbol{\Sigma}\mathbf{A}^\top)$. Use this result to obtain the distribution of the sample mean under normal random sampling. That is, let $X_1, \ldots, X_n$ be a random sample from a $N(\mu, \sigma^2)$ distribution. Find the distribution of $\overline{X}$. You might want to use $\mathbf{1}$ to represent an $n \times 1$ column vector of ones.

11. Let $X_1, \ldots, X_n$ be independent and identically distributed random variables with $E(X_i) = \mu$ and $Var(X_i) = \sigma^2$.

    (a) Show $Cov(\overline{X}, (X_j - \overline{X})) = 0$ for $j = 1, \ldots, n$.

    (b) Why does this imply that if $X_1, \ldots, X_n$ are normal, $\overline{X}$ and S^2 are independent?

12. Recall that the chi-squared distribution with ν degrees of freedom is just Gamma with $\alpha = \frac{\nu}{2}$ and $\beta = 2$. So if $X \sim \chi^2(\nu)$, it has moment-generating function $M_X(t) = (1 - 2t)^{-\nu/2}$.

    (a) Let $W_1 \sim \chi^2(\nu_1)$ and $W_2 \sim \chi^2(\nu_2)$ be independent, and $W = W_1 + W_2$. Find the distribution of W. Show your work (there's not much).

    (b) Let $W = W_1 + W_2$, where W_1 and W_2 are independent, $W \sim \chi^2(\nu_1 + \nu_2)$ and $W_2 \sim \chi^2(\nu_2)$, where ν_1 and ν_2 are both positive. Show $W_1 \sim \chi^2(\nu_1)$.

    (c) Let $X_1, \ldots, X_n$ be random sample from a $N(\mu, \sigma^2)$ distribution. Show

    $$\frac{(n-1)S^2}{\sigma^2} \sim \chi^2(n-1).$$

    Hint: $\sum_{i=1}^n (X_i - \mu)^2 = \sum_{i=1}^n \left(X_i - \overline{X} + \overline{X} - \mu\right)^2 = \ldots$

13. Let $\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\epsilon}$, where $\mathbf{X}$ is an $n \times p$ matrix of known constants, $\boldsymbol{\beta}$ is a $p \times 1$ vector of unknown constants, and $\boldsymbol{\epsilon}$ is multivariate normal with mean zero and covariance matrix $\sigma^2\mathbf{I}_n$. The constant $\sigma^2 > 0$ is unknown.

   (a) The "hat matrix" $\mathbf{H} = \mathbf{X}(\mathbf{X}^\top\mathbf{X})^{-1}\mathbf{X}^\top$.

      i. What are the dimensions (number of rows and columns) of $\mathbf{H}$?

      ii. Show $\mathbf{H}$ is symmetric.

      iii. Show $\mathbf{H}$ is "idempotent," meaning $\mathbf{HH} = \mathbf{H}$.

      iv. Show $\mathbf{I} - \mathbf{H}$ is symmetric.

      v. Show $\mathbf{I} - \mathbf{H}$ is idempotent.

   (b) What is the distribution of $\mathbf{y}$?

   (c) The least squares (and maximum likelihood) estimate of $\boldsymbol{\beta}$ is $\widehat{\boldsymbol{\beta}} = (\mathbf{X}^\top\mathbf{X})^{-1}\mathbf{X}^\top\mathbf{y}$. What is the distribution of $\widehat{\boldsymbol{\beta}}$? Show the calculations.

   (d) Let $\widehat{\mathbf{y}} = \mathbf{X}\widehat{\boldsymbol{\beta}}$.

      i. Show $\widehat{\mathbf{y}} = \mathbf{Hy}$

      ii. What is the distribution of $\widehat{\mathbf{y}}$? Show the calculation.

   (e) Let the vector of residuals $\mathbf{e} = \mathbf{y} - \widehat{\mathbf{y}}$.

      i. Show $\mathbf{e} = (\mathbf{I} - \mathbf{H})\mathbf{y}$

      ii. What is the distribution of $\mathbf{e}$? Show the calculations. Simplify both the expected value and the covariance matrix.

   (f) Using Problem 5, show that $\mathbf{e}$ and $\widehat{\boldsymbol{\beta}}$ are independent.

   (g) The least-squares (and maximum likelihood) estimator $\widehat{\boldsymbol{\beta}}$ is obtained by minimizing the sum of squares $Q = (\mathbf{y} - \mathbf{X}\boldsymbol{\beta})^\top(\mathbf{y} - \mathbf{X}\boldsymbol{\beta})$ over all $\boldsymbol{\beta} \in \mathbb{R}^p$.

      i. Show that $Q = \mathbf{e}^\top\mathbf{e} + (\widehat{\boldsymbol{\beta}} - \boldsymbol{\beta})^\top(\mathbf{X}^\top\mathbf{X})(\widehat{\boldsymbol{\beta}} - \boldsymbol{\beta})$. Hint: Add and subtract $\widehat{\mathbf{y}}$.

      ii. Why does this imply that the minimum of $Q(\boldsymbol{\beta})$ occurs at $\boldsymbol{\beta} = \widehat{\boldsymbol{\beta}}$?

      iii. The columns of $\mathbf{X}$ are linearly independent. Why does linear independence guarantee that the minimum is unique? You have just minimized a function of p variables without calculus.

      iv. Show that $W_1 = \frac{SSE}{\sigma^2} \sim \chi^2(n-p)$, where $SSE = \mathbf{e}^\top\mathbf{e}$. Use results from earlier parts of this assignment. Start with the distribution of $W = \frac{1}{\sigma^2}(\mathbf{y} - \mathbf{X}\boldsymbol{\beta})^\top(\mathbf{y} - \mathbf{X}\boldsymbol{\beta})$. This is the chi-squared random variable that appears in the denominator of all those F and t statistics. To be continued.