STA 2101/442 Assignment 1 (Mostly Review)[1]

Except for Question 5, the questions are practice for the quiz on Friday Sept. 21st, and are not to be handed in

1. The random variable X has density $f(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}$ for $0 < x < 1$, where $\alpha > 0$ and $\beta > 0$. Find $E(X)$; show your work.

2. Let $X_1, \ldots, X_n$ be a random sample from a distribution with mean μ and variance σ^2.
   (a) Show that the sample variance $S^2 = \frac{\sum_{i=1}^n (X_i - \overline{X})^2}{n-1}$ is an unbiased estimator of σ^2.
   (b) Denote the sample standard deviation by $S = \sqrt{S^2}$. Assume that the data come from a continuous distribution, so it's easy to see that $Var(S) \neq 0$. Show that S is a *biased* estimator of σ.

3. For each of the following distributions, derive a general expression for the Maximum Likelihood Estimator (MLE); don't bother with the second derivative test. Then use the data to calculate a numerical estimate; you should bring a calculator to the quiz in case you have to do something like this.
   (a) $p(x) = \theta(1-\theta)^x$ for $x = 0, 1, \ldots$, where $0 < \theta < 1$. Data: `4, 0, 1, 0, 1, 3, 2, 16, 3, 0, 4, 3, 6, 16, 0, 0, 1, 1, 6, 10`. Answer: 0.2061856
   (b) $f(x) = \frac{\alpha}{x^{\alpha+1}}$ for $x > 1$, where $\alpha > 0$. Data: `1.37, 2.89, 1.52, 1.77, 1.04, 2.71, 1.19, 1.13, 15.66, 1.43` Answer: 1.469102
   (c) $f(x) = \frac{\tau}{\sqrt{2\pi}} e^{-\frac{\tau^2 x^2}{2}}$, for x real, where $\tau > 0$. Data: `1.45, 0.47, -3.33, 0.82, -1.59, -0.37, -1.56, -0.20` Answer: 0.6451059
   (d) $f(x) = \frac{1}{\theta} e^{-x/\theta}$ for $x > 0$, where $\theta > 0$. Data: `0.28, 1.72, 0.08, 1.22, 1.86, 0.62, 2.44, 2.48, 2.96` Answer: 1.517778

4. Let $X_1, \ldots, X_n$ be a random sample from a normal distribution with mean μ and variance σ^2, so that $T = \frac{\sqrt{n}(\overline{X}-\mu)}{S} \sim t(n-1)$. This is something you don't need to prove.
   (a) Derive a $(1-\alpha)100\%$ confidence interval for μ. "Derive" means show all the high school algebra.
   (b) Show that using a t-test, $H_0 : \mu = \mu_0$ is rejected at significance level α if and only the $(1-\alpha)100\%$ confidence interval for μ does not include μ_0.

[1]This assignment was prepared by Jerry Brunner, Department of Statistics, University of Toronto. The LaTeX source code is available from the course website: `http://www.utstat.toronto.edu/~brunner/oldclass/appliedf12`

(c) Suppose that a random sample from a normal distribution yields a 95% confidence interval for μ of $(6.25, 11.34)$. Does this mean $Pr\{6.25 < \mu < 11.34\} = 0.95$? Answer Yes or No and briefly explain.

5. In Chapter One of Davison's *Statistical models*, look at the data that Charles Darwin gave to his cousin Francis Galton to analyze (Galton is responsible for the term "regression" as it is used in Statistics). Using R, do a reasonable analysis to determine whether heights of maize plants depend on the method of fertilization. Bring your printout to the quiz; you may be asked to hand it in. Be ready to

   - Describe the data set in in clear language. What are the variables? What are the cases?
   - State your model and your null hypothesis, in symbols.
   - Justify your choice of model in terms of how the data were collected.
   - Express your conclusion in plain, non-statistical language that a biologist could understand.

   Remember, the computer assignments in this course are *not group projects*. You are expected to do the work yourself. There is more than one correct answer. I did the analysis four different ways, and I consider all of them correct. I can think of about five more acceptable ways that I did not try. The number of bad ways to analyze the data is virtually unlimited.

6. Which statement is true? (Quantities in boldface are matrices of constants.)

   (a) $\mathbf{A}(\mathbf{B}+\mathbf{C}) = \mathbf{AB}+\mathbf{AC}$

   (b) $\mathbf{A}(\mathbf{B}+\mathbf{C}) = \mathbf{BA}+\mathbf{CA}$

   (c) Both a and b

   (d) Neither a nor b

7. Which statement is true?

   (a) $a(\mathbf{B}+\mathbf{C}) = a\mathbf{B}+a\mathbf{C}$

   (b) $a(\mathbf{B}+\mathbf{C}) = \mathbf{B}a+\mathbf{C}a$

   (c) Both a and b

   (d) Neither a nor b

8. Which statement is true?

   (a) $(\mathbf{B}+\mathbf{C})\mathbf{A} = \mathbf{AB}+\mathbf{AC}$
   (b) $(\mathbf{B}+\mathbf{C})\mathbf{A} = \mathbf{BA}+\mathbf{CA}$
   (c) Both a and b
   (d) Neither a nor b

9. Which statement is true?

   (a) $(\mathbf{AB})' = \mathbf{A}'\mathbf{B}'$
   (b) $(\mathbf{AB})' = \mathbf{B}'\mathbf{A}'$
   (c) Both a and b
   (d) Neither a nor b

10. Which statement is true?

    (a) $\mathbf{A}'' = \mathbf{A}$
    (b) $\mathbf{A}''' = \mathbf{A}'$
    (c) Both a and b
    (d) Neither a nor b

11. Suppose that the square matrices $\mathbf{A}$ and $\mathbf{B}$ both have inverses. Which statement is true?

    (a) $(\mathbf{AB})^{-1} = \mathbf{A}^{-1}\mathbf{B}^{-1}$
    (b) $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$
    (c) Both a and b
    (d) Neither a nor b

12. Which statement is true?

    (a) $(\mathbf{A}+\mathbf{B})' = \mathbf{A}'+\mathbf{B}'$
    (b) $(\mathbf{A}+\mathbf{B})' = \mathbf{B}'+\mathbf{A}'$
    (c) $(\mathbf{A}+\mathbf{B})' = (\mathbf{B}+\mathbf{A})'$
    (d) All of the above
    (e) None of the above

13. Which statement is true?
    (a) $(a+b)\mathbf{C} = a\mathbf{C} + b\mathbf{C}$
    (b) $(a+b)\mathbf{C} = \mathbf{C}a + \mathbf{C}b$
    (c) $(a+b)\mathbf{C} = \mathbf{C}(a+b)$
    (d) All of the above
    (e) None of the above

14. Let $\mathbf{A}$ be a square matrix with the determinant of $\mathbf{A}$ (denoted $|\mathbf{A}|$) equal to zero. What does this tell you about $\mathbf{A}^{-1}$? No proof is required here.

15. Recall that $\mathbf{A}$ symmetric means $\mathbf{A} = \mathbf{A}'$. Let $\mathbf{X}$ be an n by p matrix. Prove that $\mathbf{X}'\mathbf{X}$ is symmetric.

16. Recall that an inverse of the matrix $\mathbf{A}$ (denoted $\mathbf{A}^{-1}$) is defined by two properties: $\mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$ and $\mathbf{A}\mathbf{A}^{-1} = \mathbf{I}$. Prove that inverses are unique, as follows. Let $\mathbf{B}$ and $\mathbf{C}$ both be inverses of $\mathbf{A}$. Show that $\mathbf{B} = \mathbf{C}$.

17. Let $\mathbf{X}$ be an n by p matrix with $n \neq p$. Why is it incorrect to say that $(\mathbf{X}'\mathbf{X})^{-1} = \mathbf{X}^{-1}\mathbf{X}'^{-1}$?

18. Suppose that the square matrices $\mathbf{A}$ and $\mathbf{B}$ both have inverses. Prove that $(\mathbf{A}\mathbf{B})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$. You have two things to show.

19. Let $\mathbf{A}$ be a non-singular square matrix. Prove $(\mathbf{A}^{-1})' = (\mathbf{A}')^{-1}$.

20. Using Question 19, prove that the if the inverse of a symmetric matrix exists, it is also symmetric.

21. Let $\mathbf{a}$ be an $n \times 1$ matrix of real constants. How do you know $\mathbf{a}'\mathbf{a} \geq 0$?

22. Let $\mathbf{X}$ be an $n \times p$ matrix of constants. Recall the definition of linear independence. The columns of $\mathbf{X}$ are said to be *linearly dependent* if there exists a $p \times 1$ vector $\mathbf{v} \neq \mathbf{0}$ with $\mathbf{X}\mathbf{v} = \mathbf{0}$. We will say that the columns of $\mathbf{X}$ are linearly *independent* if $\mathbf{X}\mathbf{v} = \mathbf{0}$ implies $\mathbf{v} = \mathbf{0}$.
    (a) Show that if the columns of $\mathbf{X}$ are linearly dependent, then the columns of $\mathbf{X}'\mathbf{X}$ are also linearly dependent.
    (b) Show that if the columns of $\mathbf{X}$ are linearly dependent, then the *rows* of $\mathbf{X}'\mathbf{X}$ are linearly dependent.
    (c) Show that if the columns of $\mathbf{X}$ are linearly independent, then the columns of $\mathbf{X}'\mathbf{X}$ are also linearly independent. Use Problem 21 and the definition of linear independence.

23. Let $\mathbf{A}$ be a square matrix. Show that if the columns of $\mathbf{A}$ are linearly dependent, $\mathbf{A}^{-1}$ cannot exist. Hint: $\mathbf{v}$ cannot be both zero and not zero at the same time.

24. Recall the *spectral decomposition* of a square symmetric matrix (For example, a variance-covariance matrix). Any such matrix $\boldsymbol{\Sigma}$ can be written as $\boldsymbol{\Sigma} = \mathbf{P}\boldsymbol{\Lambda}\mathbf{P}'$, where $\mathbf{P}$ is a matrix whose columns are the (orthonormal) eigenvectors of $\boldsymbol{\Sigma}$, $\boldsymbol{\Lambda}$ is a diagonal matrix of the corresponding (non-negative) eigenvalues, and $\mathbf{P}'\mathbf{P} = \mathbf{P}\mathbf{P}' = \mathbf{I}$.

    (a) Let $\boldsymbol{\Sigma}$ be a square symmetric matrix with eigenvalues that are all strictly positive.

        i. What is $\boldsymbol{\Lambda}^{-1}$?

        ii. Show $\boldsymbol{\Sigma}^{-1} = \mathbf{P}\boldsymbol{\Lambda}^{-1}\mathbf{P}'$

    (b) Let $\boldsymbol{\Sigma}$ be a square symmetric matrix, and this time some of the eigenvalues might be zero.

        i. What do you think $\boldsymbol{\Lambda}^{1/2}$ might be?

        ii. Define $\boldsymbol{\Sigma}^{1/2}$ as $\mathbf{P}\boldsymbol{\Lambda}^{1/2}\mathbf{P}'$. Show $\boldsymbol{\Sigma}^{1/2}$ is symmetric.

        iii. Show $\boldsymbol{\Sigma}^{1/2}\boldsymbol{\Sigma}^{1/2} = \boldsymbol{\Sigma}$.

    (c) Now return to the situation where the eigenvalues of the square symmetric matrix $\boldsymbol{\Sigma}$ are all strictly positive. Define $\boldsymbol{\Sigma}^{-1/2}$ as $\mathbf{P}\boldsymbol{\Lambda}^{-1/2}\mathbf{P}'$, where the elements of the diagonal matrix $\boldsymbol{\Lambda}^{-1/2}$ are the reciprocals of the corresponding elements of $\boldsymbol{\Lambda}^{1/2}$.

        i. Show that the inverse of $\boldsymbol{\Sigma}^{1/2}$ is $\boldsymbol{\Sigma}^{-1/2}$, justifying the notation.

        ii. Show $\boldsymbol{\Sigma}^{-1/2}\boldsymbol{\Sigma}^{-1/2} = \boldsymbol{\Sigma}^{-1}$.

    (d) The (square) matrix $\boldsymbol{\Sigma}$ is said to be *positive definite* if $\mathbf{a}'\boldsymbol{\Sigma}\mathbf{a} > 0$ for all vectors $\mathbf{a} \neq \mathbf{0}$. Show that the eigenvalues of a symmetric positive definite matrix are all strictly positive. Hint: the $\mathbf{a}$ you want is an eigenvector.

    (e) Let $\boldsymbol{\Sigma}$ be a symmetric, positive definite matrix. Putting together a couple of results you have proved above, establish that $\boldsymbol{\Sigma}^{-1}$ exists.