Logistic Regression with more than two outcomes

Think of k-1 dummy variables for the dependent variable

Model for three categories

$$\ln\left(\frac{\pi_1}{\pi_3}\right) = \beta_{0,1} + \beta_{1,1}x_1 + \ldots + \beta_{p-1,1}x_{p-1}$$

$$\ln\left(\frac{\pi_2}{\pi_3}\right) = \beta_{0,2} + \beta_{1,2}x_1 + \ldots + \beta_{p-1,2}x_{p-1}$$

Need *k-1* **generalized logits** to represent a dependent variable with *k* categories

Meaning of the regression coefficients

$$\ln\left(\frac{\pi_1}{\pi_3}\right) = \beta_{0,1} + \beta_{1,1}x_1 + \ldots + \beta_{p-1,1}x_{p-1}$$

$$\ln\left(\frac{\pi_2}{\pi_3}\right) = \beta_{0,2} + \beta_{1,2}x_1 + \ldots + \beta_{p-1,2}x_{p-1}$$

A positive regression coefficient for logit *j* means that higher values of the independent variable are associated with greater chances of response category *j*, compared to the reference category.

Solve for the probabilities

$$\ln\left(\frac{\pi_1}{\pi_3}\right) = L_1 \qquad \frac{\pi_1}{\pi_3} = e^{L_1}$$

so

$$\ln\left(\frac{\pi_2}{\pi_3}\right) = L_2 \qquad \frac{\pi_2}{\pi_3} = e^{L_2}$$

So

$$\pi_1 = \pi_3 e^{L_1}$$

$$\pi_2 = \pi_3 e^{L_2}$$

Three linear equations in 3 unknowns

$$\begin{aligned}\pi_1 &= \pi_3 e^{L_1}\\ \pi_2 &= \pi_3 e^{L_2}\\ \pi_1+\pi_2+\pi_3 &= 1\end{aligned}$$

Solution

$$\begin{aligned}\pi_1 &= \frac{e^{L_1}}{1+e^{L_1}+e^{L_2}}\\ \pi_2 &= \frac{e^{L_2}}{1+e^{L_1}+e^{L_2}}\\ \pi_k &= \frac{1}{1+e^{L_1}+e^{L_2}}\end{aligned}$$

In general, solve *k* equations in *k* unknowns

$$\begin{aligned}\pi_1 &= \pi_k e^{L_1}\\ &\vdots\\ \pi_{k-1} &= \pi_k e^{L_{k-1}}\\ \pi_1+\cdots+\pi_k &= 1\end{aligned}$$

General Solution

$$\begin{aligned}\pi_1 &= \frac{e^{L_1}}{1+\sum_{j=1}^{k-1} e^{L_j}}\\ \pi_2 &= \frac{e^{L_2}}{1+\sum_{j=1}^{k-1} e^{L_j}}\\ &\vdots\\ \pi_{k-1} &= \frac{e^{L_{k-1}}}{1+\sum_{j=1}^{k-1} e^{L_j}}\\ \pi_k &= \frac{1}{1+\sum_{j=1}^{k-1} e^{L_j}}\end{aligned}$$

Using the solution, one can

- Calculate the probability of obtaining the observed data as a function of the regression coefficients: Get maximum likelihood estimates (*b* values)
- From maximum likelihood estimates, get tests and confidence intervals
- Using *b* values in L_j, estimate probabilities of category membership for any set of x values.