

Logistic Regression with more than two outcomes

This slide show is a free open source document.
See the last slide for copyright information.

If there are k outcomes

Think of $k-1$ dummy variables for
the response variable

Model for three categories

$$\ln \left(\frac{\pi_1}{\pi_3} \right) = \beta_{0,1} + \beta_{1,1}x_1 + \dots + \beta_{p-1,1}x_{p-1}$$

$$\ln \left(\frac{\pi_2}{\pi_3} \right) = \beta_{0,2} + \beta_{1,2}x_1 + \dots + \beta_{p-1,2}x_{p-1}$$

Need $k-1$ **generalized logits** to represent a response variable with k categories

Meaning of the regression coefficients

$$\ln \left(\frac{\pi_1}{\pi_3} \right) = \beta_{0,1} + \beta_{1,1}x_1 + \dots + \beta_{p-1,1}x_{p-1}$$

$$\ln \left(\frac{\pi_2}{\pi_3} \right) = \beta_{0,2} + \beta_{1,2}x_1 + \dots + \beta_{p-1,2}x_{p-1}$$

A positive regression coefficient for logit j means that higher values of the explanatory variable are associated with greater chances of response category j , compared to the reference category.

Solve for the probabilities

$$\ln \left(\frac{\pi_1}{\pi_3} \right) = L_1 \quad \text{so} \quad \frac{\pi_1}{\pi_3} = e^{L_1}$$

$$\ln \left(\frac{\pi_2}{\pi_3} \right) = L_2 \quad \frac{\pi_2}{\pi_3} = e^{L_2}$$

$$\pi_1 = \pi_3 e^{L_1}$$

So

$$\pi_2 = \pi_3 e^{L_2}$$

Three linear equations in 3 unknowns

$$\pi_1 = \pi_3 e^{L_1}$$

$$\pi_2 = \pi_3 e^{L_2}$$

$$\pi_1 + \pi_2 + \pi_3 = 1$$

Solution

$$\pi_1 = \frac{e^{L_1}}{1 + e^{L_1} + e^{L_2}}$$

$$\pi_2 = \frac{e^{L_2}}{1 + e^{L_1} + e^{L_2}}$$

$$\pi_k = \frac{1}{1 + e^{L_1} + e^{L_2}}$$

In general, solve k equations
in k unknowns

$$\begin{aligned}\pi_1 &= \pi_k e^{L_1} \\ &\vdots \\ \pi_{k-1} &= \pi_k e^{L_{k-1}} \\ \pi_1 + \cdots + \pi_k &= 1\end{aligned}$$

General Solution

$$\begin{aligned}\pi_1 &= \frac{e^{L_1}}{1 + \sum_{j=1}^{k-1} e^{L_j}} \\ \pi_2 &= \frac{e^{L_2}}{1 + \sum_{j=1}^{k-1} e^{L_j}} \\ &\vdots \\ \pi_{k-1} &= \frac{e^{L_{k-1}}}{1 + \sum_{j=1}^{k-1} e^{L_j}} \\ \pi_k &= \frac{1}{1 + \sum_{j=1}^{k-1} e^{L_j}}\end{aligned}$$

Using the solution, one can

- Calculate the probability of obtaining the observed data as a function of the regression coefficients: Get maximum likelihood estimates (b values)
- From maximum likelihood estimates, get tests and confidence intervals
- Using b values in L_j , estimate probabilities of category membership for any set of x values.

Copyright Information

This slide show was prepared by Jerry Brunner, Department of Statistical Sciences, University of Toronto. It is licensed under a Creative Commons Attribution - ShareAlike 3.0 Unported License. Use any part of it as you like and share the result freely. These Powerpoint slides are available from the course website:

<http://www.utstat.toronto.edu/~brunner/oldclass/441s16>