STA431 Assignment Nine

① (a) $y = C^T z \sim N_k(0, C^T \Sigma C)$

$$= N_k(0, C^T C D C^T C) = N_k(0, D)$$

(b) $Var(y_j) = \lambda_j$

(c) Because zero covariance implies independence for the multivariate normal.

(d) $y = C^T z \iff z = Cy$

(e) i) $z_1 = c_{11} y_1 + c_{12} y_2 + \cdots + c_{1k} y_k$

ii) $Var(z_1) = c_{11}^2 \lambda_1 + c_{12}^2 \lambda_2 + \cdots + c_{1k}^2 \lambda_k$

iii) $c_{14}^2 \lambda_4$

(f) $cov(z, y) = cov(z, C^T z) = cov(z z) C$

$$= \Sigma C = C D C^T C = CD$$

(g) $CD = [c_{ij} \lambda_j]$

(h) $Corr(z_1, y_4)^2 = \left(\frac{cov(z_1, y_4)}{1 \cdot \sqrt{\lambda_4}} \right)^2$

$$= \left(\frac{c_{14} \lambda_4}{\sqrt{\lambda_4}} \right)^2 = c_{14}^2 \lambda_4$$

Same as (eiii)

(1i) $\text{cov}(y_2) = \text{cov}(D^{-\frac{1}{2}} y) = \text{cov}(D^{-\frac{1}{2}} C^T z)$

$= D^{-\frac{1}{2}} C^T \Sigma C D^{-\frac{1}{2}} = D^{-\frac{1}{2}} C^T C D C^T C D^{-\frac{1}{2}} = \underline{I}$

(j) $y_2 = D^{-\frac{1}{2}} y = D^{-\frac{1}{2}} C^T z$

$\Leftrightarrow z = C D^{\frac{1}{2}} y_2$

(k) $\text{cov}(z, y_2) = \text{cov}(C D^{\frac{1}{2}} y_2, y_2)$

$= C D^{\frac{1}{2}} \text{cov}(y_2) = C D^{\frac{1}{2}}$

(l) Want diagonal of $\text{cov}(z, y_2)^T \text{cov}(z, y_2)$

$= (C D^{\frac{1}{2}})^T C D^{\frac{1}{2}} = D^{\frac{1}{2}} C^T C D^{\frac{1}{2}} = D$

The eigenvalues.

(2) $f = J y_2$

(a) $cov(f) = cov(J y_2) = J\, cov(y_2)\, J^T$

$= J I_k J^T = J J^T = I_p$ As given in the question

(b) $cov(z, f) = cov(z, J y_2) = cov(z, y_2) J^T$

$= C D^{\frac{1}{2}} J^T = L$

↑ From (9k)

(c) $f' = Rf$

(i) $cov(f') = Cov(Rf) = R\, cov(f) R^T$

$= R I_p R^T = I_p$

(ii) $cov(z, f') = cov(z, Rf) = cov(z, f) R^T$

$= L R^T$ Correlations because both random vectors are standardized.

(iii) $cov(z, f) = L$ and $cov(z, f') = L R^T$

$diag(L L^T) = diag(L R^T R L^T)$

$= diag(L R^T (L R^T)^T)$

The "operation" is to post- multiply the matrix by it's transpose

(3) (a) 0.730^2 [illegible]

= 0.533

(b) 0.330^2 [illegible]

= 0.109

(c) $0.763^2 + (-0.201)^2 + 0.186^2$

$= 0.657$

(d) $0.718^2 + 0.257^2 + (-0.274)^2$

$= 0.657$

(4)

(a) $Var(e_{i2}) = 1 - \lambda_{21}^2 - \lambda_{22}^2$

(b) $1 - \lambda_{21}^2 - \lambda_{22}^2$

(c) $\lambda_{21}^2 + \lambda_{22}^2$

(d) $Corr(z_{i3}, F_{i2}) = cov(z_{i3}, F_{i2})$

$= cov(\lambda_{31}F_{i1} + \lambda_{32}F_{i2}, F_{i2}) = cov(\lambda_{32}F_{i2}, F_{i2})$

$= \lambda_{32} \, var(F_{i2}) = \lambda_{32}$

(e) λ_{32}^2

(f) $Corr(s_i, F_{i2})^2$

$= \left(\frac{cov(s_i, F_{i2})}{\sqrt{var(s_i)} \sqrt{1}} \right)^2$

$= \frac{(\lambda_{11} + \lambda_{21} + \lambda_{31} + \lambda_{41} + \lambda_{51})^2}{5}$

(See how any negative factor loadings will make reliability smaller

(g) $\lambda_{41}^2 + \lambda_{42}^2$

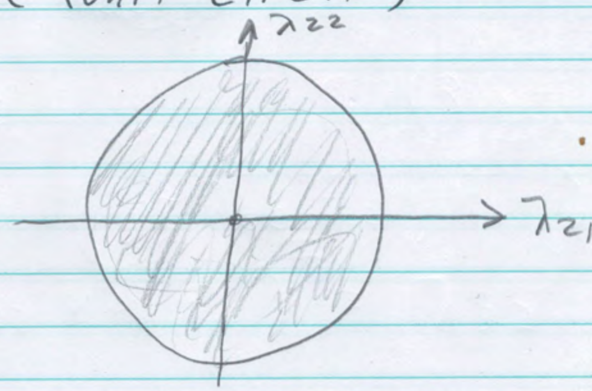
(h) $\frac{1}{5}(\lambda_{11}^2 + \lambda_{21}^2 + \lambda_{31}^2 + \lambda_{41}^2 + \lambda_{51}^2)$

(4, i) $Corr(z_{i2}, z_{i5}) = \lambda_{21}\lambda_{51} + \lambda_{22}\lambda_{52}$

(j) $\theta = (\lambda_{11}\ \lambda_{12}\ \lambda_{21}\ \lambda_{22}\ \lambda_{31}\ \lambda_{32}\ \lambda_{41}\ \lambda_{42}\ \lambda_{51}\ \lambda_{52})$

(k) $\lambda_{21}^2 + \lambda_{22}^2 \leq 1$

(l) A disc (unit circle)



(m) There are $5(5-1)/2 = 10$ correlations and $5 \times 2 = 10$ parameters : Yes

(5) (a) $\mathrm{cov}(d_i, F_i) = \mathrm{cov}(\Lambda F_i + e_i, F_i) = \Lambda\,\mathrm{cov}(F_i) =$
$= \Lambda \Phi$

(b) $\mathrm{cov}(d_i) = \mathrm{cov}(\Lambda F_i + e_i) = \Lambda \Phi \Lambda^T + \Omega$

(c) $\Sigma = \mathrm{cov}(d_i) = \Lambda \Phi \Lambda^T + \Omega$

$$= \Lambda \Phi^{\frac{1}{2}} \Phi^{\frac{1}{2}} \Lambda^T + \Omega$$

$$= (\Lambda \Phi^{\frac{1}{2}}) I_p (\Lambda \Phi^{\frac{1}{2}})^T + \Omega$$

$$= \Lambda_2 I_p \Lambda_2^T + \Omega$$

So there are two different parameter sets, (Λ, Φ, Ω) and (Λ_2, I_p, Ω) that yield the same Σ.

(d) $\Sigma = \Lambda \Lambda^T + \Omega = \Lambda R^T R \Lambda^T + \Omega$

$$= (\Lambda R^T)(\Lambda R^T)^T + \Omega = \Lambda_3 \Lambda_3^T + \Omega$$

Again, we have two distinct parameter sets, (Λ, Ω) and (Λ_3, Ω) yielding the same Σ.

(e) (i) $\mathrm{Cov}(z_i, F_i) = \mathrm{cov}(\Lambda F_i + e_i, F_i) = \Lambda\,\mathrm{cov}(F_i) = \Lambda$

$\mathrm{Cov}(z_i\ F_i') = \mathrm{cov}(\Lambda_2 F_i' + e_i, F_i') = \Lambda_2\,\mathrm{cov}(F_i')$

$= \Lambda_2\,\mathrm{cov}(RF) = \Lambda_2 R I R^T = \Lambda_2$

ii) $\Lambda_2 \Lambda_2^T = \Lambda R^T (\Lambda R^T)^T = \Lambda R^T R \Lambda^T = \Lambda \Lambda^T$

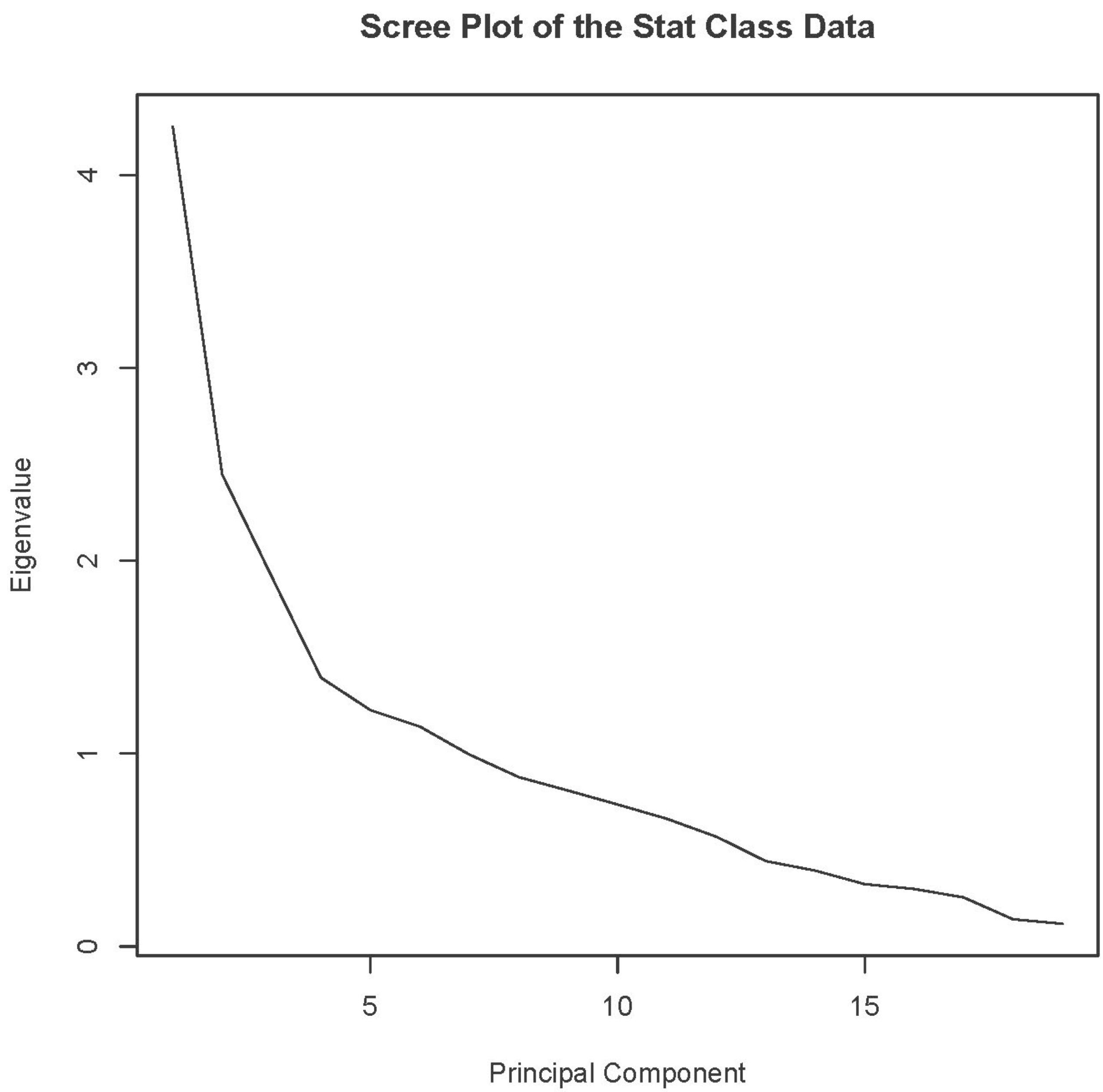
Matrices are equal so diagonals are equal.

Assignment 9, Question 6

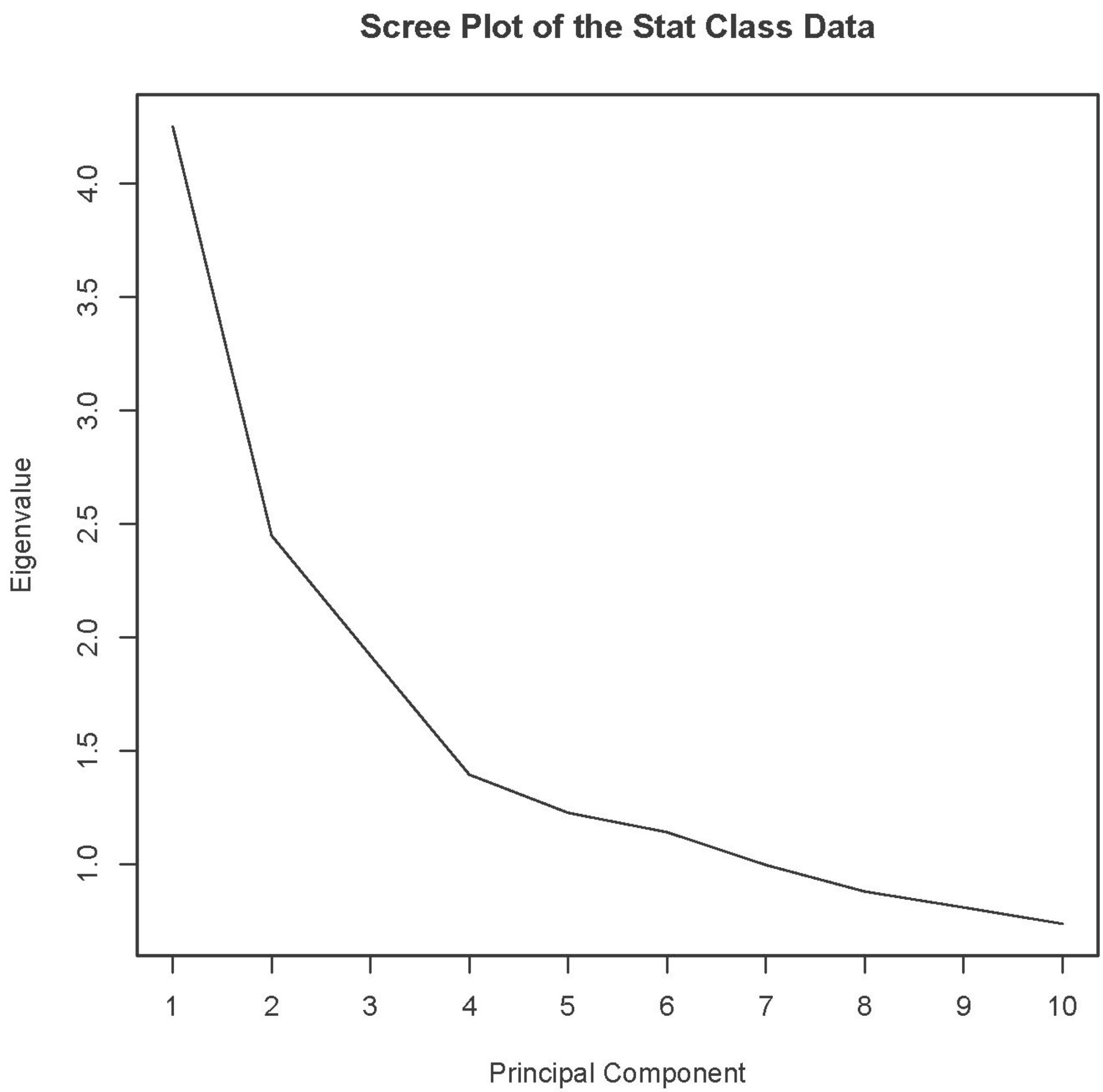
```
> # EFA of the statclass data
>
> rm(list=ls())
> statclass =
read.table("https://www.utstat.toronto.edu/brunner/openSEM/data/statclass.data.txt",
header=TRUE)
> # dim(statclass); head(statclass)
> dat = statclass[,3:21]; dim(dat); head(dat)
[1] 58 19
  Q1 Q2 Q3 Q4 Q5 Q6 Q7 Q8 C1 C2 C3 C4 C5 C6 C7 C8 C9 MT Final
1  9  1  7  8  4  3  5  2  6 10 10 10  5  0  0  0  0 55    43
2 10 10  5  9 10  8  6  8 10 10  8  9  9  9  9 10 10 66    79
3 10 10  5 10 10 10  9  8 10 10 10 10 10 10  9 10 10 94    67
4 10 10  8  9 10  7 10  9 10 10 10  9 10 10  9 10 10 81    65
5 10  6  7  9  8  8  5  7 10  9 10  9  5  6  4  8 10 57    52
6 10  9  5  8  9  8  5  6  8  7  5  6 10  6  5  9  9 77    64

> # (a) How many factors?
>
> pc = prcomp(dat, scale = T)
> Eigenvalue = pc$sdev^2 ; Eigenvalue
 [1] 4.2527781 2.4483506 1.9187278 1.3940113 1.2256366 1.1402548 0.9955204 0.8785114
 [9] 0.8083985 0.7363114 0.6619296 0.5688996 0.4426665 0.3932991 0.3232235 0.2984006
[17] 0.2544390 0.1409714 0.1176700
> # Could be 6
```

```
> # Scree plot
> plot(1:19,Eigenvalue, xlab = "Principal Component", type = "l",
+      main = "Scree Plot of the Stat Class Data")
```



```
> # Zoom in on the first ten
> plot(1:10,Eigenvalue[1:10], type = "l",
+     xaxp = c(1,10,9), # Tick marks on x axis
+     xlab = "Principal Component", ylab = "Eigenvalue",
+      main = "Scree Plot of the Stat Class Data")
```



```
> # help(factanal)
> fa4 = factanal(dat,factors=4) # rotation="varimax" is the default
> fa4

Call:
factanal(x = dat, factors = 4)

Uniquenesses:
   Q1    Q2    Q3    Q4    Q5    Q6    Q7    Q8    C1    C2    C3    C4    C5    C6 
0.918 0.527 0.855 0.121 0.795 0.534 0.525 0.805 0.666 0.523 0.005 0.823 0.324 0.284 
   C7    C8    C9    MT Final 
0.542 0.374 0.945 0.790 0.432 

Loadings:
      Factor1 Factor2 Factor3 Factor4
Q1     0.189  -0.113          -0.177 
Q2     0.654           0.208         
Q3     0.169           0.334         
Q4             0.203   0.909         
Q5     0.120   0.406  -0.156         
Q6     0.594   0.237   0.199   0.131 
Q7     0.482   0.179   0.193   0.415 
Q8     0.431                         
C1     0.345   0.151   0.143   0.414 
C2                     0.666   0.178 
C3    -0.170  -0.111   0.535   0.817 
C4            -0.140   0.385         
C5    -0.119   0.788   0.196         
C6             0.724   0.159   0.398 
C7     0.347   0.389           0.429 
C8     0.496   0.571           0.227 
C9     0.105   0.203                 
MT     0.401   0.211                 
Final  0.740          -0.114         

               Factor1 Factor2 Factor3 Factor4
SS loadings      2.536   2.083   2.069   1.523
Proportion Var   0.133   0.110   0.109   0.080
Cumulative Var   0.133   0.243   0.352   0.432

Test of the hypothesis that 4 factors are sufficient.
The chi square statistic is 102.1 on 101 degrees of freedom.
The p-value is 0.451 
```

```
> L4 = fa4$loadings; print(L4,cutoff=0.3, sort="True")

Loadings:
      Factor1 Factor2 Factor3 Factor4
Q2     0.654                         
Q6     0.594                         
Final  0.740                         
C5             0.788                 
C6             0.724           0.398 
C8     0.496   0.571                 
Q4                     0.909         
C2                     0.666         
C3                     0.535   0.817 
Q1                                   
Q3                     0.334         
Q5             0.406                 
Q7     0.482                   0.415 
Q8     0.431                         
C1     0.345                   0.414 
C4                     0.385         
C7     0.347   0.389           0.429 
C9                                   
MT     0.401                         

               Factor1 Factor2 Factor3 Factor4
SS loadings      2.536   2.083   2.069   1.523
Proportion Var   0.133   0.110   0.109   0.080
Cumulative Var   0.133   0.243   0.352   0.432
```

```
>
> # Try 3 factors
> fa3 = factanal(dat,factors=3)
> print(fa3,cutoff=0.3, sort="True")

Call:
factanal(x = dat, factors = 3)

Uniquenesses:
   Q1    Q2    Q3    Q4    Q5    Q6    Q7    Q8    C1    C2    C3    C4    C5    C6
0.943 0.554 0.853 0.472 0.808 0.516 0.575 0.777 0.751 0.386 0.568 0.840 0.637 0.090
   C7    C8    C9    MT Final
0.577 0.381 0.942 0.796 0.565

Loadings:
      Factor1 Factor2 Factor3
Q2     0.654
Q6     0.662
Q7     0.519
C8     0.591   0.513
Final  0.642
C5             0.584
C6             0.916
C7     0.393   0.519
Q4                     0.718
C2                     0.777
C3                     0.626
Q1
Q3
Q5             0.353
Q8     0.467
C1     0.423
C4                     0.398
C9
MT     0.424

               Factor1 Factor2 Factor3
SS loadings      2.795   2.131   2.044
Proportion Var   0.147   0.112   0.108
Cumulative Var   0.147   0.259   0.367

Test of the hypothesis that 3 factors are sufficient.
The chi square statistic is 137.51 on 117 degrees of freedom.
The p-value is 0.0946
```

```
>
> # Try 2 factors
> fa2 = factanal(dat,factors=2)
> print(fa2,cutoff=0.3, sort="True")

Call:
factanal(x = dat, factors = 2)

Uniquenesses:
   Q1    Q2    Q3    Q4    Q5    Q6    Q7    Q8    C1    C2    C3    C4    C5    C6
1.000 0.761 0.924 0.479 0.842 0.594 0.619 0.826 0.734 0.417 0.564 0.842 0.830 0.571
   C7    C8    C9    MT Final
0.638 0.329 0.926 0.833 0.781

Loadings:
      Factor1 Factor2
Q6     0.606
Q7     0.509   0.350
C6     0.548   0.359
C7     0.589
C8     0.818
Q4             0.722
C2             0.764
C3             0.660
Q1
Q2     0.467
Q3
Q5     0.380
Q8     0.415
C1     0.467
C4             0.386
C5     0.331
C9
MT     0.406
Final  0.449

               Factor1 Factor2
SS loadings      3.257   2.234
Proportion Var   0.171   0.118
Cumulative Var   0.171   0.289

Test of the hypothesis that 2 factors are sufficient.
The chi square statistic is 187.03 on 134 degrees of freedom.
The p-value is 0.0017

> #  (d) I guess I will settle for 4.

> #  (e) Get communalities 2 different ways, and use cbind to display them
> #      side by side.
> ls(fa4)

 [1] "call"          "converged"     "correlation"  "criteria"      "dof"
 [6] "factors"       "loadings"      "method"       "n.obs"         "PVAL"
[11] "rotmat"        "STATISTIC"     "uniquenesses"
```

```
> commun = diag(L4 %*% t(L4) )
> cbind(commun, 1-fa4$uniquenesses)

          commun
Q1    0.08240676 0.08240312
Q2    0.47306931 0.47307067
Q3    0.14456256 0.14456023
Q4    0.87859474 0.87859464
Q5    0.20510501 0.20511097
Q6    0.46588629 0.46588706
Q7    0.47453865 0.47453844
Q8    0.19479587 0.19479909
C1    0.33373977 0.33374219
C2    0.47704598 0.47704520
C3    0.99500540 0.99500000
C4    0.17678212 0.17678212
C5    0.67617032 0.67616986
C6    0.71553982 0.71553934
C7    0.45838861 0.45838767
C8    0.62609996 0.62610066
C9    0.05544739 0.05543382
MT    0.20960873 0.20961304
Final 0.56785720 0.56785778

> # (f, g) See output above.

> # (h)
>
> print(L4,cutoff=0.3, sort="True")

Loadings:
      Factor1 Factor2 Factor3 Factor4
Q2     0.654
Q6     0.594
Final  0.740
C5             0.788
C6             0.724           0.398
C8     0.496   0.571
Q4                     0.909
C2                     0.666
C3                     0.535   0.817
Q1
Q3                     0.334
Q5             0.406
Q7     0.482                   0.415
Q8     0.431
C1     0.345                   0.414
C4                     0.385
C7     0.347   0.389           0.429
C9
MT     0.401

               Factor1 Factor2 Factor3 Factor4
SS loadings      2.536   2.083   2.069   1.523
Proportion Var   0.133   0.110   0.109   0.080
Cumulative Var   0.133   0.243   0.352   0.432

> # (i) Estimated reliability of the final exam as a measure of Factor 1
> 0.740^2
[1] 0.5476
```