STA 431 Assignment 8

1

(a)

	Endogenous	Exogenous
Observable	Y_1 Y_2	X
Latent		ε_1 ε_2

(b) Independently for $i = 1, \ldots, n$ let

$$Y_{i1} = \gamma_0 + \gamma_1 X_i + \varepsilon_{i1}$$

$$Y_{i2} = \beta_0 + \beta_1 Y_{i1} + \varepsilon_{i2} \quad \text{where}$$

$X_i \sim N(\mu_x, \phi)$, $\varepsilon_{i1} \sim N(0, \psi_1)$, $\varepsilon_{i2} \sim N(0, \psi_2)$

and X_i, ε_{i1} & ε_{i2} are independent.

(c) $\theta = (\gamma_0, \gamma_1, \beta_0, \beta_1, \mu_x, \phi, \psi_1, \psi_2)$

$$E(X) = \mu_x$$

(d) $$E(Y_1) = E(\gamma_0 + \gamma_1 X + \varepsilon_1)$$
$$= \gamma_0 + \gamma_1 \mu_x$$

$$E(Y_2) = E(\beta_0 + \beta_1 Y_1 + \varepsilon_2)$$

$$= \beta_0 + \beta_1 E(Y_1) + 0$$

$$= \beta_0 + \beta_1(\gamma_0 + \gamma_1 \mu_x)$$

$$= \beta_0 + \beta_1 \gamma_0 + \beta_1 \gamma_1 \mu_x$$

(1d continued)

$$Y_1 = \gamma_0 + \gamma_1 X + \varepsilon_1$$
$$Y_2 = \beta_0 + \beta_1 Y_1 + \varepsilon_2$$
$$= \beta_0 + \beta_1 \gamma_0 + \gamma_1 \beta_1 X + \beta_1 \varepsilon_1 + \varepsilon_2$$

$$\Sigma = \begin{array}{c|c|c} X & Y_1 & Y_2 \\ \hline \phi & \gamma_1 \phi & \gamma_1 \beta_1 \phi \\ \hline & \gamma_1^2 \phi + \psi_1 & \beta_1(\gamma_1^2 \phi + \psi_1) \\ \hline & & \gamma_1^2 \beta_1^2 \phi + \beta_1^2 \psi_1 + \psi_2 \end{array}$$

$$\mathrm{cov}(Y_1, Y_2) = \mathrm{cov}(Y_1, \beta_0 + \beta_1 Y_1 + \varepsilon_2)$$
$$= \beta_1 \,\mathrm{var}(Y_1) = \beta_1(\gamma_1^2 \phi + \psi_1)$$

$$\mu = \begin{pmatrix} \mu_x \\ \gamma_0 + \gamma_1 \mu_x \\ \beta_0 + \beta_1(\gamma_0 + \gamma_1 \mu_x) \end{pmatrix}$$

(1e) Again, $\theta = (\gamma_0, \gamma_1, \beta_0, \beta_1, \mu_x, \phi, \psi_1, \psi_2)$

$\phi = \sigma_{11}$

$\gamma_1 = \sigma_{12} / \phi$

$\beta_1 = \sigma_{13} / (\gamma_1 \cancel{\phi})$ That's $\sigma_{13} / (\gamma_1 \phi)$

$\psi_1 = \sigma_{22} - \gamma_1^2 \phi$

$\psi_2 = \sigma_{33} - \gamma_1^2 \beta_1^2 \phi - \beta_1^2 \psi_1$

$\mu_x = \mu_1$ That's μ_1

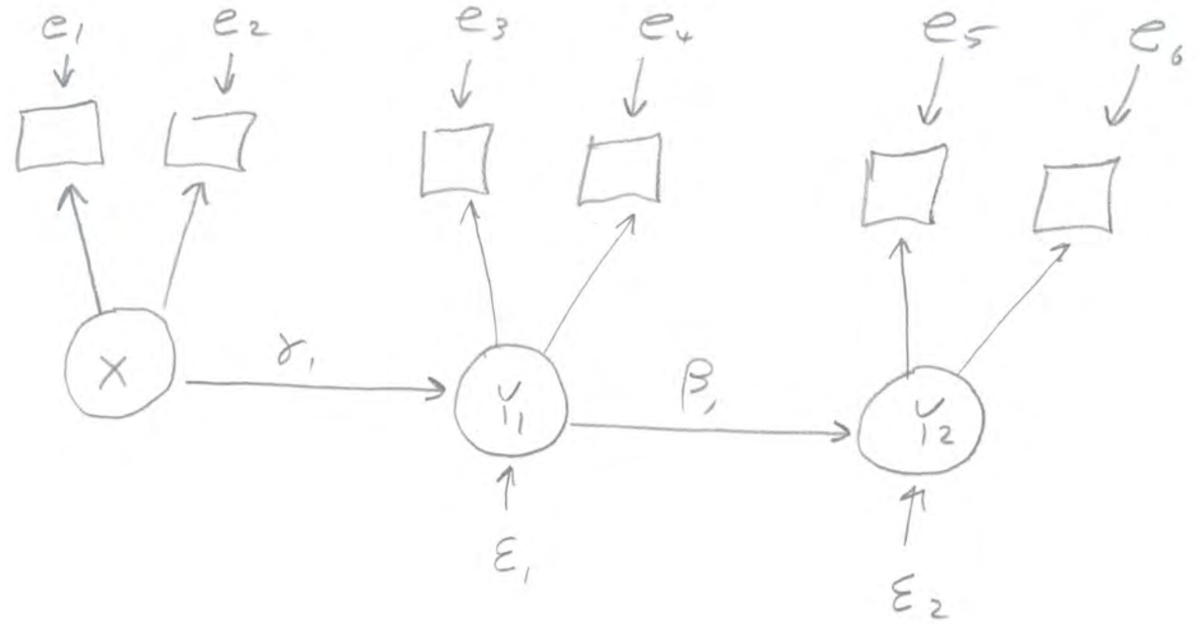
$\gamma_0 = \mu_2 - \gamma_1 \mu_x$

$\beta_0 = \mu_3 - \beta_1 (\gamma_0 + \gamma_1 \mu_x)$

(f) No. There are 8 parameters and 9 moment structure equations.

(1g)

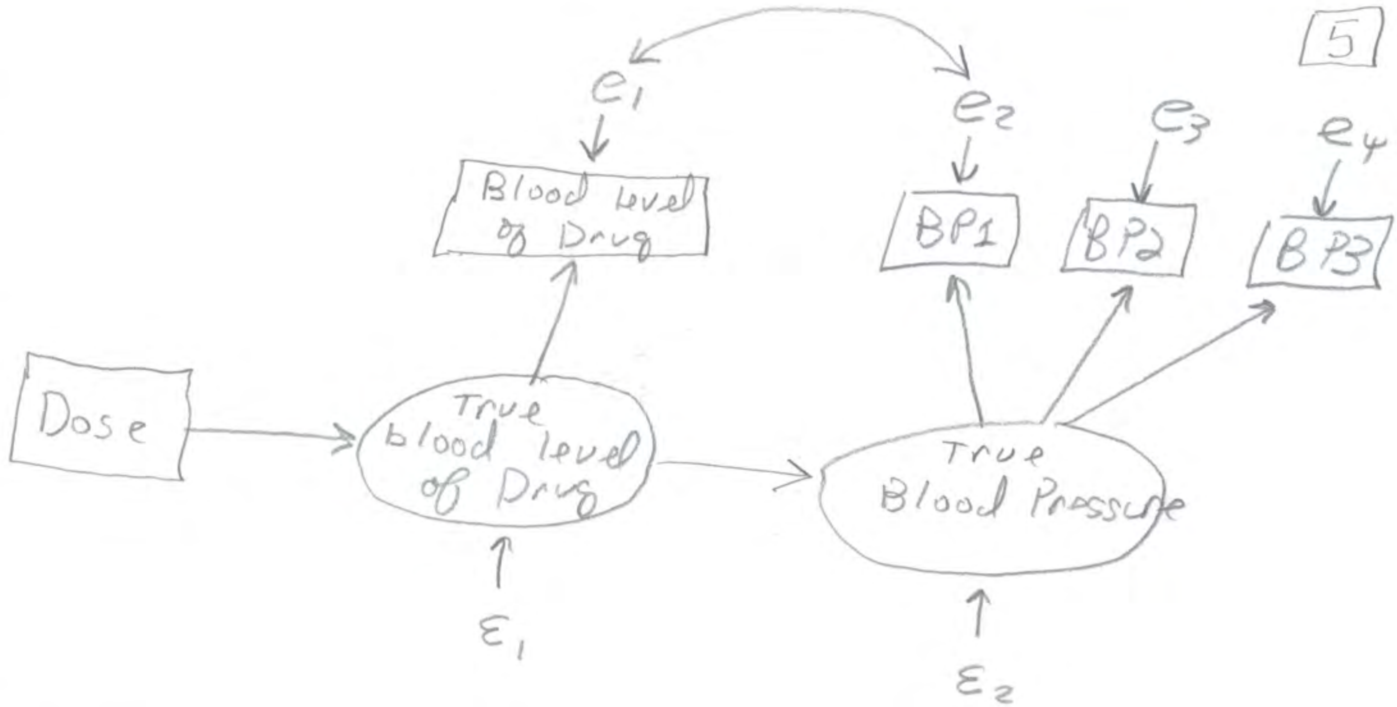
(i)



(ii)

Yes, all parameters are identifiable. Identifiability of the parameters in the latent model was established in part (e) of this question – and anyway it's established by the Acyclic Rule. Parameters of the measurement model are identifiable by the Double Measurement Rule, and parameters of the combined model are identifiable by the Two-Stage Rule.

(2)



(3) (a) Model Equations in scalar form

$$Y_1 = \gamma_1 X_1 + \beta_1 Y_3 + \varepsilon_1$$

$$Y_2 = \gamma_2 X_2 + \beta_2 Y_1 + \varepsilon_2$$

$$Y_3 = \gamma_3 X_1 + \gamma_3 X_2 + \beta_3 Y_2 + \varepsilon_3$$

$$D_1 = \lambda_1 X_1 + e_1$$

$$D_2 = \lambda_1 X_1 + e_2$$

$$D_3 = \lambda_2 Y_1 + e_3$$

$$D_4 = \lambda_3 Y_2 + e_4$$

$$D_5 = \lambda_4 X_2 + e_5$$

$$D_6 = \lambda_4 X_2 + e_6$$

(3b.)

$$\begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \end{pmatrix} = \begin{pmatrix} 0 & 0 & \beta_1 \\ \beta_2 & 0 & 0 \\ 0 & \beta_3 & 0 \end{pmatrix} \begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \end{pmatrix} + \begin{pmatrix} \gamma_1 & 0 \\ 0 & \gamma_2 \\ \gamma_3 & \gamma_3 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} + \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \end{pmatrix}$$

(c)

$$\begin{pmatrix} D_1 \\ D_2 \\ D_3 \\ D_4 \\ D_5 \\ D_6 \end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 & 0 & 0 & 0 \\ \lambda_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & 0 & \lambda_3 & 0 \\ 0 & \lambda_4 & 0 & 0 & 0 \\ 0 & \lambda_4 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \\ Y_1 \\ Y_2 \\ Y_3 \end{pmatrix} + \begin{pmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \\ e_6 \end{pmatrix}$$

$$d = \Lambda \quad F + e$$

(d)

$$\Phi = (\emptyset)$$

$$\Psi = \begin{pmatrix} \psi_1 & 0 & 0 \\ 0 & \psi_2 & 0 \\ 0 & 0 & \psi_3 \end{pmatrix}$$

(3d continued)

$$\Omega = \begin{array}{c|cccccc} & e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \\ \hline e_1 & w_{11} & w_{12} & 0 & 0 & 0 & 0 \\ e_2 & & w_{22} & 0 & 0 & 0 & 0 \\ e_3 & & & w_{33} & 0 & 0 & 0 \\ e_4 & & & & w_{44} & w_{45} & 0 \\ e_5 & & & & & w_{55} & w_{56} \\ e_6 & & & & & & w_{66} \end{array}$$

It's symmetric of course.

4 (a)

(4) $y_i = \beta y_i + \Gamma x_i + \varepsilon_i$

(a) $y_i - \beta y_i = \Gamma x_i + \varepsilon_i$

$\Rightarrow I y_i - \beta y_i = \Gamma x_i + \varepsilon_i$

$\Rightarrow (I - \beta) y_i = \Gamma x_i + \varepsilon_i$

$\Rightarrow y_i = (I-\beta)^{-1} \Gamma x_i + (I-\beta)^{-1} \varepsilon_i$

So $\operatorname{cov}(y_i) = (I-\beta)^{-1} \Gamma \operatorname{cov}(x_i) \left((I-\beta)^{-1}\Gamma\right)^T + (I-\beta)^{-1} \Psi (I-\beta)^{-1T}$

$$= (I-\beta)^{-1} \Gamma \Phi_x \Gamma^T \left((I-\beta)^T\right)^{-1} + (I-\beta)^{-1} \Psi (I-\beta^T)^{-1}$$

$$= (I-\beta)^{-1} \Gamma \Phi_x \Gamma^T (I-\beta^T)^{-1} + (I-\beta)^{-1} \Psi (I-\beta^T)^{-1}$$

(b) $\operatorname{cov}(x_i, y_i) = \operatorname{cov}(x_i, (I-\beta)^{-1} \Gamma x_i)$

$$= \operatorname{cov}(x_i, x_i) \left((I-\beta)^{-1} \Gamma\right)^T$$

$$= \Phi_x \Gamma^T (I-\beta^T)^{-1}$$

(5) (a)

$$(I-\beta)y_i = \Gamma x_i + \varepsilon_i$$

$$\Rightarrow v^T(I-\beta)y_i = v^T\Gamma x_i + v^T\varepsilon_i$$

$$\Rightarrow 0 = v^T\Gamma x_i + v^T\varepsilon_i$$

$$\Rightarrow v^T\varepsilon_i = -v^T\Gamma x_i$$

(b) $\Rightarrow \text{cov}(v^T\varepsilon_i) = -\text{cov}(v^T\Gamma x_i)$

$$\Rightarrow v^T\Psi v = -v^T\Gamma\Phi_x\Gamma^T v \qquad (*)$$

$v^T\Psi v > 0$ because Ψ is positive definite and $v^T\Gamma\Phi_x\Gamma^T v \geq 0$ ~~because~~ because $\text{cov}(\Gamma x_i) = \Gamma\Phi_x\Gamma^T$ is a covariance matrix, and all covariance matrices are non-negative definite

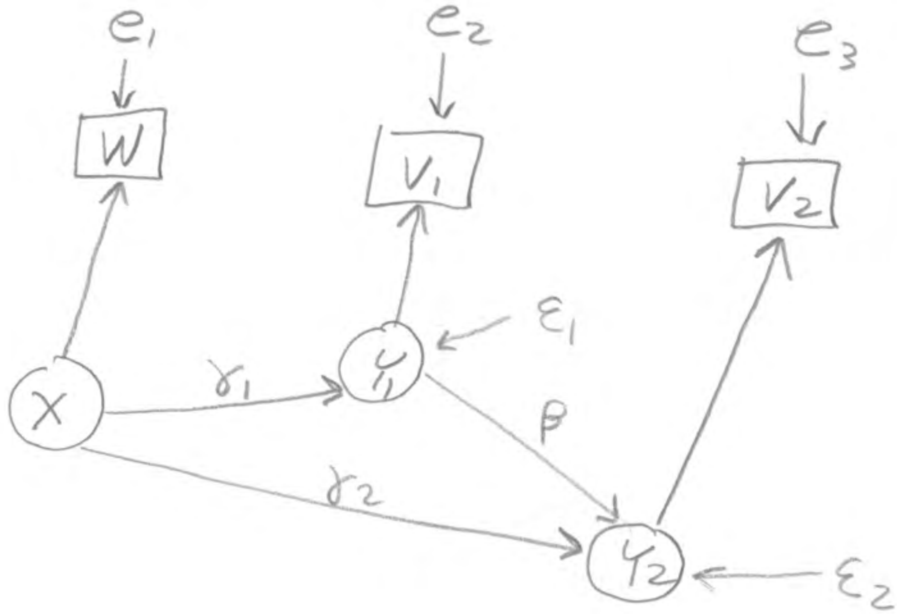
Thus, $0 < v^T\Psi v = -v^T\Gamma\Phi_x\Gamma^T v \leq 0$, a contradiction. The existence of such a v is impossible. Thus the rows of $(I-\beta)$ are linearly independent, and $(I-\beta)^{-1}$ exists ∎

Another way to produce a contradiction is from (*)

$$\underset{\substack{\| \\ v^T\Psi v > 0}}{\text{cov}(v^T\varepsilon_i, v^T\varepsilon_i)} = \underset{\substack{\| \\ 0}}{\text{cov}(v^T\varepsilon_i, -v^T\Gamma x_i)}$$

(6)

(a)



(b) No. $\theta = (\phi \; \gamma_1 \; \gamma_2 \; \beta \; \psi_1 \; \psi_2 \; \omega_1 \; \omega_2 \; \omega_3)$

9 parameters and 6 covariance structure equations. Fails the parameter count rule

(7) (a) Independently for $i=1,\ldots,n$, let

$$D_{i,1} = F_i + e_{i,1}$$
$$D_{i,2} = \lambda F_i + e_{i,2}$$
$$D_{i,3} = \lambda F_i + e_{i,3}$$

where
$Var(F_i) = \phi$
$Var(e_{i,1}) = \omega_1$
$Var(e_{i,2}) = \omega_2$
$Var(e_{i,3}) = \omega_3$

(b) $\theta = (\lambda, \phi, \omega_1, \omega_2, \omega_3)$

(c) Yes. 6 equations in 5 unknown parameters.

(d)
$$\Sigma = \begin{array}{c|ccc} & D_1 & D_2 & D_3 \\ \hline D_1 & \phi+\omega_1 & \lambda\phi & \lambda\phi \\ D_2 & & \lambda^2\phi+\omega_2 & \lambda^2\phi \\ D_3 & & & \lambda^2\phi+\omega_3 \end{array}$$

If $\lambda=0$, then λ, ω_2 and ω_3 are identifiable but ϕ and ω_1 are not. If $\lambda \neq 0$,

$\phi = \sigma_{12}\sigma_{13}/\sigma_{23}$, $\lambda = \sigma_{13}/\phi$, $\omega_1 = \sigma_{11} - \phi$
$\omega_2 = \sigma_{22} - \lambda^2\phi$, $\omega_3 = \sigma_{33} - \lambda^2\phi$
And all parameters are identifiable.

(e) 6 equations in 5 unknowns: One df

8

(a)
$$Y_1 = \gamma X + \varepsilon_1$$
$$Y_2 = \beta_1 Y_1 + \varepsilon_2$$
$$Y_3 = \beta_2 Y_1 + \varepsilon_3$$
$$V = Y_1 + e$$

(b) $\theta = (\gamma \;\; \beta_1 \;\; \beta_2 \;\; \phi \;\; \omega \;\; \psi_1 \;\; \psi_2 \;\; \psi_3 \;\; \psi_{23})$

(c) Yes $4(4+1)/2 = 10$ equations in 9 unknown parameters

(d) Just part of the covariance matrix:

	X	V	Y_2	Y_3
X	ϕ	$\gamma\phi$	$\gamma\beta_1\phi$	$\gamma\beta_2\phi$

$$\phi = \sigma_{11}$$
$$\gamma = \sigma_{12}/\sigma_{11}$$
$$\beta_1 = \sigma_{13}/\sigma_{12}$$
$$\beta_2 = \sigma_{14}/\sigma_{12}$$

Showing identifiability provided β_1, β_2 & $\gamma \neq 0$

$\boxed{13.5}$

$Y_1 = \gamma X + \varepsilon_1$

$Y_2 = \beta_1 Y_1 + \varepsilon_2 = \beta_1(\gamma X + \varepsilon_1) + \varepsilon_2$

$= \gamma\beta_1 X + \beta_1\varepsilon_1 + \varepsilon_2$

$Y_3 = \gamma\beta_2 X + \beta_2\varepsilon_1 + \varepsilon_3$

$V = Y_1 + e = \gamma X + \varepsilon_1 + e$

$\Sigma =$

$$
\begin{array}{c|c|c|c|c|}
 & X & V & Y_2 & Y_3 \\
\hline
X & \phi & \gamma\phi & \gamma\beta_1\phi & \gamma\beta_2\phi \\
\hline
V & & \gamma^2\phi + \psi_1 + \omega & \gamma^2\beta_1\phi + \beta_1\psi_1 & \gamma^2\beta_2\phi + \beta_2\psi_1 \\
\hline
Y_2 & & & \gamma^2\beta_1^2\phi + \beta_1^2\psi_1 + \psi_2 & \gamma^2\beta_1\beta_2\phi + \beta_1\beta_2\psi_1 + \psi_{23} \\
\hline
Y_3 & & & & \gamma^2\beta_2^2\phi + \beta_2^2\psi_1 + \psi_3 \\
\hline
\end{array}
$$

$\theta = (\gamma \;\; \phi \;\; \psi_1 \;\; \beta_1 \;\; \psi_2 \;\; \beta_2 \;\; \psi_3 \;\; \omega \;\; \psi_{23})$

Assignment 8, Question 9

```
> # Simulate manipulation check design
> # Assignment 8, Question 9
>
> rm(list=ls())
> # source("rmvn.txt")
> source("https://www.utstat.toronto.edu/~brunner/openSEM/fun/rmvn.txt")
>
> # install.packages("lavaan", dependencies = TRUE) # Only need to do this once
> library(lavaan)

This is lavaan 0.6-11
lavaan is FREE software! Please report any bugs.
>
> # Set parameter values and sample size
> gamma = 1; beta1 = 0.75; beta2 = 1.25; phi = 2
> omega = 1; psi1 = 3; psi2 = 4; psi3 = 5; psi23 = 2
> n = 10000
>
> ################## Generate data ##################
> set.seed(888999)
> X = rnorm(n,0,sqrt(phi))
> epsilon1 = rnorm(n,0,sqrt(psi1))
> e = rnorm(n,0,sqrt(omega))
> V23 = rbind(c(psi2,psi23),
+             c(psi23,psi3))
> er = rmvn(n,c(0,0),V23)
> epsilon2 = er[,1]
> epsilon3 = er[,2]
>
> Y1 = gamma*X + epsilon1
> Y2 = beta1*Y1 + epsilon2
> Y3 = beta2*Y1 + epsilon3
> V = Y1 + e
> Mcheck = cbind(X,V,Y2,Y3)
> #####################################################
>
> cor(Mcheck)
           X         V        Y2        Y3
X  1.0000000 0.5726864 0.3931787 0.4845121
V  0.5726864 1.0000000 0.5790716 0.7084621
Y2 0.3931787 0.5790716 1.0000000 0.7146340
Y3 0.4845121 0.7084621 0.7146340 1.0000000
```

```
> ####### Fit model with lavaan
>
> mod1 = 'Y1 ~ gamma*X
+         Y1 =~ 1.0*V + beta1*Y2 + beta2*Y3
+         # Variances and covariances
+         X ~~ phi*X     # Var(X) = phi
+         V ~~ omega*V   # Var(e) = omega
+         Y1 ~~ psi1*Y1  # Var(epsilon1) = psi1
+         Y2 ~~ psi2*Y2  # Var(epsilon2) = psi2
+         Y3 ~~ psi3*Y3  # Var(epsilon3) = psi3
+         Y2 ~~ psi23*Y3 # Cov(epsilon2,epsilon3) = psi23
+        ' # End of mod1
>
> fit1 = lavaan(mod1, data=Mcheck)
> # summary(fit1)
> parameterEstimates(fit1)
   lhs op rhs label   est    se      z pvalue ci.lower ci.upper
1   Y1  ~   X gamma 0.983 0.014 69.859      0    0.955    1.010
2   Y1 =~   V       1.000 0.000     NA     NA    1.000    1.000
3   Y1 =~  Y2 beta1 0.732 0.013 56.490      0    0.707    0.758
4   Y1 =~  Y3 beta2 1.247 0.018 68.087      0    1.211    1.283
5    X ~~   X   phi 2.010 0.028 70.711      0    1.954    2.066
6    V ~~   V omega 0.956 0.060 15.817      0    0.838    1.075
7   Y1 ~~  Y1  psi1 3.020 0.073 41.500      0    2.877    3.162
8   Y2 ~~  Y2  psi2 3.997 0.068 58.848      0    3.864    4.130
9   Y3 ~~  Y3  psi3 5.164 0.118 43.867      0    4.933    5.395
10  Y2 ~~  Y3 psi23 2.088 0.075 27.934      0    1.941    2.234

>
> # Try mod2 with just one measurement of Y1
> # and regressions connecting Y1 to Y2 and Y3
>
> mod2 = 'Y1 ~ gamma*X
+         Y2 ~ beta1*Y1
+         Y3 ~ beta2*Y1
+         Y1 =~ 1.0*V
+         # Variances and covariances
+         X ~~ phi*X     # Var(X) = phi
+         V ~~ omega*V   # Var(e) = omega
+         Y1 ~~ psi1*Y1  # Var(epsilon1) = psi1
+         Y2 ~~ psi2*Y2  # Var(epsilon2) = psi2
+         Y3 ~~ psi3*Y3  # Var(epsilon3) = psi3
+         Y2 ~~ psi23*Y3 # Cov(epsilon2,epsilon3) = psi23
+        ' # End of mod2
>
> fit2 = lavaan(mod2, data=Mcheck)
> parameterEstimates(fit2)
   lhs op rhs label   est    se      z pvalue ci.lower ci.upper
1   Y1  ~   X gamma 0.983 0.014 69.859      0    0.955    1.010
2   Y2  ~  Y1 beta1 0.732 0.013 56.490      0    0.707    0.758
3   Y3  ~  Y1 beta2 1.247 0.018 68.087      0    1.211    1.283
4   Y1 =~   V       1.000 0.000     NA     NA    1.000    1.000
5    X ~~   X   phi 2.010 0.028 70.711      0    1.954    2.066
6    V ~~   V omega 0.956 0.060 15.817      0    0.838    1.075
7   Y1 ~~  Y1  psi1 3.020 0.073 41.500      0    2.877    3.162
8   Y2 ~~  Y2  psi2 3.997 0.068 58.848      0    3.864    4.130
9   Y3 ~~  Y3  psi3 5.164 0.118 43.867      0    4.933    5.395
10  Y2 ~~  Y3 psi23 2.088 0.075 27.934      0    1.941    2.234
```