STA431s23 Assignment 7

(1)

(a)

	W_1	V_1	W_2	V_2
W_1	$\Phi_x + \Omega_{11}$	$\Phi_x \beta^T + \Omega_{12}$	Φ_x	$\Phi_x \beta^T$
V_1		$\beta \Phi_x \beta^T + \Psi + \Omega_{22}$	$\beta \Phi_x$	$\beta \Phi_x \beta^T + \Psi$
W_2			$\Phi_x + \Omega_{33}$	$\Phi_x \beta^T + \Omega_{34}$
V_2				$\beta \Phi_x \beta^T + \Psi + \Omega_{44}$

$$cov(W_1, V_1) = cov(x+e_1, y+e_2) = cov(x,y) + cov(e_1, e_2)$$
$$= cov(x, \beta x + \epsilon) + \Omega_{12} = \Phi_x \beta^T + \Omega_{12}$$

$$cov(W_1, V_2) = cov(x+e_1, y+e_4) = cov(x,y) + 0 = \Phi_x \beta^T$$
$$cov(V_1, V_1) = cov(y+e_2, y+e_2) = cov(y) + cov(e_2) = \beta \Phi_x \beta^T + \Omega_{22}$$
$$cov(V_1, V_2) = cov(y+e_2, y+e_4) = cov(y) + 0 = \beta \Phi_x \beta^T + \Psi$$

$$cov(V_1, W_2) = cov(y+e_1, x+e_3) = cov(y, x) = \beta \Phi_x$$

$$cov(W_2) = cov(x+e_3) = \Phi_x + \Omega_{33}$$

(1b) $\underset{\substack{\uparrow \\ \text{unique elements of}}}{\theta =} (\Phi_x \ \Omega_{11} \ \beta \ \Omega_{12} \ \Omega_{22} \ \Psi \ \Omega_{33} \ \Omega_{34} \ \Omega_{44})$

$$\Phi_x = \Sigma_{13}$$

$$\Omega_{11} = \Sigma_{11} - \Phi_x$$

$$\beta = \Sigma_{23} \Phi_x^{-1}$$

$$\Omega_{12} = \Sigma_{12} - \Phi_x \beta^T$$

$$\Psi = \Sigma_{24} - \beta \Phi_x \beta^T$$

$$\Omega_{22} = \Sigma_{22} - \beta \Phi_x \beta^T - \Psi$$

$$\Omega_{33} = \Sigma_{33} - \Phi_x$$

$$\Omega_{34} = \Sigma_{34} - \Phi_x \beta^T$$

$$\Omega_{44} = \Sigma_{44} - \beta \Phi_x \beta^T - \Psi$$

(c) $\tilde{\Phi}_x = \frac{1}{2}(\hat{\Sigma}_{13} + \hat{\Sigma}_{13}^T)$ To make it symmetric

(d) $\tilde{\beta}_n = \frac{1}{2}(\hat{\Sigma}_{23} + \hat{\Sigma}_{14}^T)\tilde{\Phi}_x^{-1} = (\hat{\Sigma}_{23} + \hat{\Sigma}_{14}^T)(\hat{\Sigma}_{13} + \hat{\Sigma}_{13}^T)^{-1}$

$\xrightarrow{p} (\Sigma_{23} + \Sigma_{14}^T)(\Sigma_{13} + \Sigma_{13}^T)^{-1}$ by $\hat{\Sigma} \xrightarrow{p} \Sigma$ and continuous mappings

$\overset{''}{=} (\beta\Phi_x + \beta\Phi_x)(\Phi_x + \Phi_x)^{-1} = 2\beta\Phi_x(2\Phi_x)^{-1} = 2\beta\Phi_x \cdot \frac{1}{2}\Phi_x^{-1}$

$= \beta \Phi_x \Phi_x^{-1} = \beta$

(2) (a) Φ_x is $p \times p$: $p(p+1)/2$

Ω_{11} is $p \times p$: $p(p+1)/2$

β is $q \times p$: pq

Ω_{12} is $p \times q$: pq

Ω_{22} is $q \times q$: $q(q+1)/2$

Ψ is $q \times q$: $q(q+1)/2$

Ω_{33} is $p \times p$: $p(p+1)/2$

Ω_{34} is $p \times q$: pq

Ω_{44} is $q \times q$: $q(q+1)/2$

$$3\left(p(p+1)/2 + q(q+1)/2 + pq \right)$$

(b) Σ is $2(p+q)$ by $2(p+q)$, so number of unique elements is $2(p+q)(2p+2q+1)/2$
$= (p+q)(2p+2q+1)$

(c) Looking at the answer to (1a), $\Sigma_{14} = \Sigma_{23}^T$, so that's pq. Also Σ_{13} is symmetric, so that's $p(p-1)/2$ more. Σ_{24} is symmetric, for $q(q-1)/2$ more. Total number of constraints is $pq + p(p-1)/2 + q(q-1)/2$

(d) Number of covariance structure equations minus number of parameters is $(p+q)(2p+2q+1) - \frac{3}{2}\left(p(p+1) + q(q+1) + 2pq\right)$

$$= 2p^2 + 2pq + p + 2pq + 2q^2 + q - \frac{3}{2}p^2 - \frac{3}{2}p - \frac{3}{2}q^2 - \frac{3}{2}q - 3pq$$

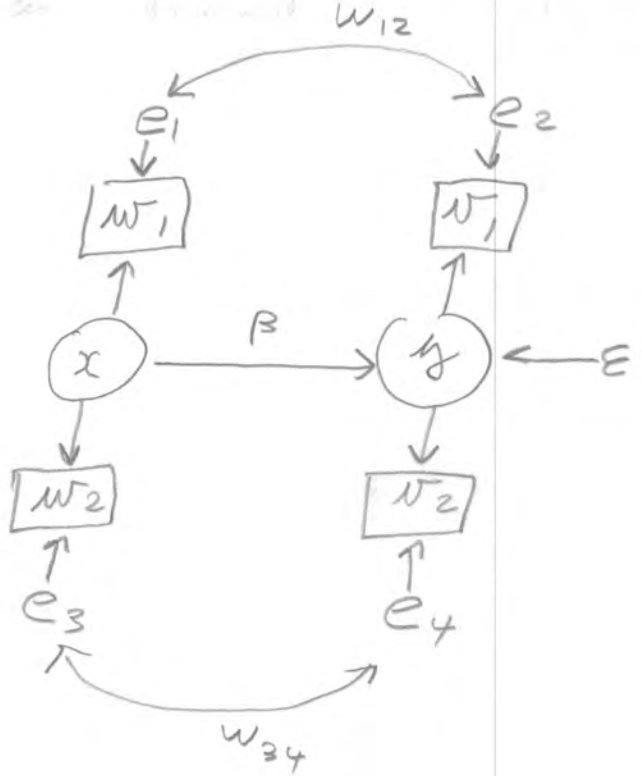
$$= p^2\left(2 - \frac{3}{2}\right) + pq + (p+q)(1 - 3/2) + q^2\left(2 - \frac{3}{2}\right)$$

$$= \frac{1}{2}p^2 + \frac{1}{2}q^2 - \frac{1}{2}(p+q) + pq = \frac{1}{2}(p^2 - p) + \frac{1}{2}(q^2 - q) + pq$$

$$= pq + p(p-1)/2 + q(q-1)/2 = \text{Number of constraints!}$$

(3) (a) and (b) are on the printout

(c)



w_1 = n breeders Q1

v_1 = n givebirth Q1

w_2 = n breeders Q2

v_2 = n givebirth Q2

x = Number of breeders

y = Number giving birth

(d) Independently for $i = 1, \ldots, n$

$$y_i = \beta x_i + \epsilon_i$$

$$w_{i1} = x_i + e_{i1}$$

$$v_{i1} = y_i + e_{i2}$$

$$w_{i2} = x_i + e_{i3}$$

$$v_{i2} = y_i + e_{i4}$$

where all expected values are zero (optional)

$Var(x_i) = \phi \quad Var(\epsilon_i) = \psi$

$Var(e_{i1}) = \omega_{11} \quad Var(e_{i2}) = \omega_{22} \quad Cov(e_{i1}, e_{i2}) = \omega_{12}$

$Var(e_{i3}) = \omega_{33} \quad Var(e_{i4}) = \omega_{44} \quad Cov(e_{i3}, e_{i4}) = \omega_{34}$

And all other covariances are zero.

(3e) On the printout.

(f) Yes. $G^2 = 0.087$, $df=1$, $p=0.768$
The df agrees with question 2. With $p=q=1$,
$$df = pq + p(p-1)/2 + q(q-1)/2 = 1+0+0 = 1$$

(g) $\hat{\beta} = 0.757$

(h) See printout. $\tilde{\beta} = 0.764$ is pretty close to $\hat{\beta}$.

(i) See printout: $(0.651, 0.862)$

(j) See rel1 and rel2 on printout

(k) See printout

(l) Yes. See z-tests for $\omega_{12} \neq \omega_{34}$

(m) (i) $H_0: \omega_{11} = \omega_{33}$ and $\omega_{33} = \omega_{44}$

(ii) $W_n = 23.93$, $df = 2$, $p = 0.00000635$

(iii) Measurement of number of sows giving birth appears to be more accurate for Questionnaire 2. See printout.

Assignment 7, Question 3

```
> # a.
> pigs = read.table("http://www.utstat.toronto.edu/brunner/openSEM/data/openpigs2.data.txt")
> head(pigs); dim(pigs)
  nbreedersQ1 ngivebirthQ1 nbreedersQ2 ngivebirthQ2
1          69           45          89           67
2          52           24          85           41
3          48           33          68           34
4           7            4          30           30
5          41           37          54           35
6          35           23          57           48
[1] 114   4
>
> # b.
> S = var(pigs); S
             nbreedersQ1 ngivebirthQ1 nbreedersQ2 ngivebirthQ2
nbreedersQ1     691.3607     581.6943    348.5299     272.6710
ngivebirthQ1    581.6943     562.6853    260.0286     239.2279
nbreedersQ2     348.5299     260.0286    770.9685     370.8930
ngivebirthQ2    272.6710     239.2279    370.8930     334.1683

> # c. and d. are handwritten.
>
> # e.
>
> # install.packages("lavaan", dependencies = TRUE) # Only need to do this once
> library(lavaan)
This is lavaan 0.6-11
lavaan is FREE software! Please report any bugs.
>
> pigmod =
+    '#########################################################
+     # Latent variable model
+     # ---------------------
+     Lbirth ~ beta*Lbreeders
+     #
+     # Measurement model
+     # -----------------
+     Lbreeders =~ 1*nbreedersQ1  + 1*nbreedersQ2
+     Lbirth    =~ 1*ngivebirthQ1 + 1*ngivebirthQ2
+     #
+     # Variances
+     # ---------
+     Lbreeders ~~ phi*Lbreeders          # Var(Lbreeders) = phi
+     Lbirth ~~ psi*Lbirth                # Var(epsilon) = psi
+     nbreedersQ1  ~~ omega1*nbreedersQ1  # Var(e1) = omega1
+     ngivebirthQ1 ~~ omega2*ngivebirthQ1 # Var(e2) = omega2
+     nbreedersQ2  ~~ omega3*nbreedersQ2  # Var(e3) = omega3
+     ngivebirthQ2 ~~ omega4*ngivebirthQ2 # Var(e4) = omega4
+     #
+     # Covariances (between error terms)
+     # ---------------------------------
+     nbreedersQ1  ~~ omega12*ngivebirthQ1  # Cov(e1,e2) = omega12
+     nbreedersQ2  ~~ omega34*ngivebirthQ2  # Cov(e3,e4) = omega34
+     # Reliabilities of number of breeding sows for the two questionnaires
+     # -----------------------------------------------------------------
+     rel1 := omega1/(phi+omega1)
+     rel2 := omega3/(phi+omega3)
+    ' #################### End of pigmod ######################
>
> fit1 = lavaan(pigmod, data=pigs)
> summary(fit1)
```

```
lavaan 0.6-11 ended normally after 120 iterations

  Estimator                                         ML
  Optimization method                           NLMINB
  Number of model parameters                         9

  Number of observations                           114

Model Test User Model:

  Test statistic                                 0.087
  Degrees of freedom                                 1
  P-value (Chi-square)                           0.768

Parameter Estimates:

  Standard errors                             Standard
  Information                                 Expected
  Information saturated (h1) model          Structured

Latent Variables:
                   Estimate  Std.Err  z-value  P(>|z|)
  Lbreeders =~
    nbreedersQ1       1.000
    nbreedersQ2       1.000
  Lbirth =~
    ngivebirthQ1      1.000
    ngivebirthQ2      1.000

Regressions:
                   Estimate  Std.Err  z-value  P(>|z|)
  Lbirth ~
    Lbredrs (beta)    0.757    0.054   14.047    0.000

Covariances:
                   Estimate  Std.Err  z-value  P(>|z|)
 .nbreedersQ1 ~~
   .ngvbrQ1 (om12)  308.539   57.249    5.389    0.000
 .nbreedersQ2 ~~
   .ngvbrQ2 (om34)  101.406   45.513    2.228    0.026

Variances:
                   Estimate  Std.Err  z-value  P(>|z|)
    Lbredrs  (phi)  357.145   64.936    5.500    0.000
   .Lbirth   (psi)   33.634   10.861    3.097    0.002
   .nbrdrQ1 (omg1)  330.683   67.114    4.927    0.000
   .ngvbrQ1 (omg2)  321.255   54.160    5.932    0.000
   .nbrdrQ2 (omg3)  416.335   80.763    5.155    0.000
   .ngvbrQ2 (omg4)   93.000   35.566    2.615    0.009

Defined Parameters:
                   Estimate  Std.Err  z-value  P(>|z|)
    rel1              0.481    0.070    6.877    0.000
    rel2              0.538    0.070    7.706    0.000

>
> # h.
>
> # MOM of beta: Really need to calculate Sigma to see this.
> 0.5*(S[1,4]+S[2,3])/S[1,3]
[1] 0.7642093
```

```
>
> # i.
> parameterEstimates(fit1)
            lhs op                 rhs   label     est     se      z pvalue ci.lower ci.upper
1        Lbirth  ~           Lbreeders    beta   0.757  0.054 14.047  0.000    0.651    0.862
2     Lbreeders =~        nbreedersQ1            1.000  0.000     NA     NA    1.000    1.000
3     Lbreeders =~        nbreedersQ2            1.000  0.000     NA     NA    1.000    1.000
4        Lbirth =~       ngivebirthQ1            1.000  0.000     NA     NA    1.000    1.000
5        Lbirth =~       ngivebirthQ2            1.000  0.000     NA     NA    1.000    1.000
6     Lbreeders ~~           Lbreeders     phi 357.145 64.936  5.500  0.000  229.873  484.417
7        Lbirth ~~              Lbirth     psi  33.634 10.861  3.097  0.002   12.348   54.920
8   nbreedersQ1 ~~         nbreedersQ1  omega1 330.683 67.114  4.927  0.000  199.143  462.224
9  ngivebirthQ1 ~~        ngivebirthQ1  omega2 321.255 54.160  5.932  0.000  215.103  427.407
10  nbreedersQ2 ~~         nbreedersQ2  omega3 416.335 80.763  5.155  0.000  258.042  574.628
11 ngivebirthQ2 ~~        ngivebirthQ2  omega4  93.000 35.566  2.615  0.009   23.291  162.709
12  nbreedersQ1 ~~        ngivebirthQ1 omega12 308.539 57.249  5.389  0.000  196.334  420.744
13  nbreedersQ2 ~~        ngivebirthQ2 omega34 101.406 45.513  2.228  0.026   12.201  190.610
14         rel1 := omega1/(phi+omega1)    rel1   0.481  0.070  6.877  0.000    0.344    0.618
15         rel2 := omega3/(phi+omega3)    rel2   0.538  0.070  7.706  0.000    0.401    0.675

> # j. k. and L.  See above.
>
> # m. Compare precision of measurement.
> # H0: omega1=omega3 and omega2=omega4
> # For Wald tests: Wtest = function(L,Tn,Vn,h=0) # H0: L theta = h
> source("http://www.utstat.utoronto.ca/~brunner/Rfunctions/Wtest.txt")
>
> # Fit model with bootstrap SEs
> fit2 = lavaan(pigmod, data=pigs, se = "bootstrap")
> summary(fit2) # I didn't ask for this explicitly, so no questions on it.
lavaan 0.6-11 ended normally after 120 iterations

  Estimator                                         ML
  Optimization method                           NLMINB
  Number of model parameters                         9

  Number of observations                           114

Model Test User Model:

  Test statistic                                 0.087
  Degrees of freedom                                 1
  P-value (Chi-square)                           0.768

Parameter Estimates:

  Standard errors                            Bootstrap
  Number of requested bootstrap draws             1000
  Number of successful bootstrap draws            1000

Latent Variables:
                   Estimate  Std.Err  z-value  P(>|z|)
  Lbreeders =~
    nbreedersQ1       1.000
    nbreedersQ2       1.000
  Lbirth =~
    ngivebirthQ1      1.000
    ngivebirthQ2      1.000
```

```
Regressions:
                   Estimate  Std.Err  z-value  P(>|z|)
  Lbirth ~
    Lbredrs (beta)    0.757    0.071   10.721    0.000

Covariances:
                   Estimate  Std.Err  z-value  P(>|z|)
 .nbreedersQ1 ~~
   .ngvbrQ1 (om12)  308.539   54.937    5.616    0.000
 .nbreedersQ2 ~~
   .ngvbrQ2 (om34)  101.406   37.255    2.722    0.006

Variances:
                   Estimate  Std.Err  z-value  P(>|z|)
    Lbredrs  (phi)  357.145   81.988    4.356    0.000
   .Lbirth   (psi)   33.634   11.319    2.972    0.003
   .nbrdrQ1 (omg1)  330.683   60.901    5.430    0.000
   .ngvbrQ1 (omg2)  321.255   52.548    6.114    0.000
   .nbrdrQ2 (omg3)  416.335   86.653    4.805    0.000
   .ngvbrQ2 (omg4)   93.000   30.053    3.095    0.002

Defined Parameters:
                   Estimate  Std.Err  z-value  P(>|z|)
    rel1              0.481    0.085    5.686    0.000
    rel2              0.538    0.082    6.537    0.000

>
> # Wald test of H0: omega1=omega3 and omega2=omega4
> That = coef(fit2); That
   beta     phi     psi  omega1  omega2  omega3  omega4 omega12 omega34
  0.757 357.145  33.634 330.683 321.255 416.335  93.000 308.539 101.406
> VV = vcov(fit2)
> LL = rbind(c(0, 0, 0, 1, 0,-1, 0, 0, 0),
+            c(0, 0, 0, 0, 1, 0,-1, 0, 0))
> Wtest(LL,That,VV)
           W           df      p-value
2.393441e+01 2.000000e+00 6.349046e-06
>
> # Follow-up
> # omega1 vs. omega3
> L13 = rbind(c(0,  0,  0,  1,  0,-1,  0,  0,  0))
> Wtest(L13,That,VV)
        W        df   p-value
0.7262899 1.0000000 0.3940884
> # omega2 vs. omega4
> L24 = rbind(c(0,  0,  0,  0,  1,  0,-1,  0,  0))
> Wtest(L24,That,VV)
           W           df      p-value
1.338262e+01 1.000000e+00 2.539661e-04
>
```

4 (a) C_{23} is $q \times p$

(b) i. C has $p+q+p+q = 2p+2q$ rows & columns, so number of unique elements is $2(p+q)(2p+2q+1)/2$
$= (p+q)(2p+2q+1)$

ii. Φ_z has $p(p+1)/2$ unique elements

iii. K has p^2 elements

(c) Σ is $(2p+q) \times (2p+q)$, so there are $(2p+q)(2p+q+1)/2$ unique elements.

(d) No way. There are fewer covariance structure equations than elements of C..

(e) Under the naive model $v = \beta w + \varepsilon$

$$\Sigma = cov\begin{pmatrix} z \\ w \\ v \end{pmatrix} = \left[\begin{array}{c|c|c} \Phi_z & K & K\beta^T \\ \hline & \Phi_w & \Phi_w \beta^T \\ \hline & & \beta\Phi_w\beta^T + \Psi \end{array}\right]$$

$$cov(z, v) = cov(z, \beta w + \varepsilon) = cov(z, \beta(x+e) + \varepsilon)$$
$$= cov(z, \beta x) + 0 = K\beta^T$$

$$cov(w, v) = cov(w, \beta w + \varepsilon) = \Phi_w \beta^T$$

$$cov(v) = cov(\beta w + \varepsilon, \beta w + \varepsilon) = \beta\Phi_w\beta^T + \Psi$$

So $\beta^T = \Sigma_{22}^{-1}\Sigma_{23}$, and $\hat{\beta}_{bad} = \hat{\Sigma}_{23}^{T}\hat{\Sigma}_{22}^{-1} = \hat{\Sigma}_{32}\hat{\Sigma}_{22}^{-1}$

(4f) $\hat{\beta}_{bad} = \hat{\Sigma}_{32}\hat{\Sigma}_{22}^{-1} \xrightarrow{p} \Sigma_{32}\Sigma_{22}^{-1}$

$$= cov(v, w)\, cov(w)^{-1}$$

Under the true model,

$$cov(v, w) = cov(y+e_2, x+e_1) = cov(\beta x + \varepsilon + e_2, x+e_1)$$

$$= \beta cov(x) + \beta cov(x, e_1) + cov(\varepsilon, x) + cov(\varepsilon, e_1) + cov(e_2, x) + cov(e_2 e_1)$$

$$= \beta \Phi_x + \beta C_{13} + C_{12}^T + C_{23} + C_{14}^T + C_{34}^T$$

$$= \beta\Phi_x + \beta C_{13} + C_{21} + C_{23} + C_{41} + C_{43}$$

And

$$cov(w) = cov(x+e_1, x+e_1) = cov(x,x) + cov(x, e_1) + cov(e_1, x) + cov(e_1)$$

$$= \Phi_x + C_{13} + C_{31} + \Omega_1$$

So $\hat{\beta}_{bad} \xrightarrow{p} (\beta\Phi_x + \beta C_{13} + C_{21} + C_{23} + C_{41} + C_{43})(\Phi_x + C_{13} + C_{31} + \Omega_1)^{-1}$

(g) How about $\hat{\beta}_{good} = \hat{\Sigma}_{31}\hat{\Sigma}_{21}^{-1} \xrightarrow{p} \Sigma_{31}\Sigma_{21}^{-1}$

$$= cov(v, z)\, cov(w, z)^{-1} = cov(y+e_2, z)\, cov(x+e_1, z)^{-1}$$

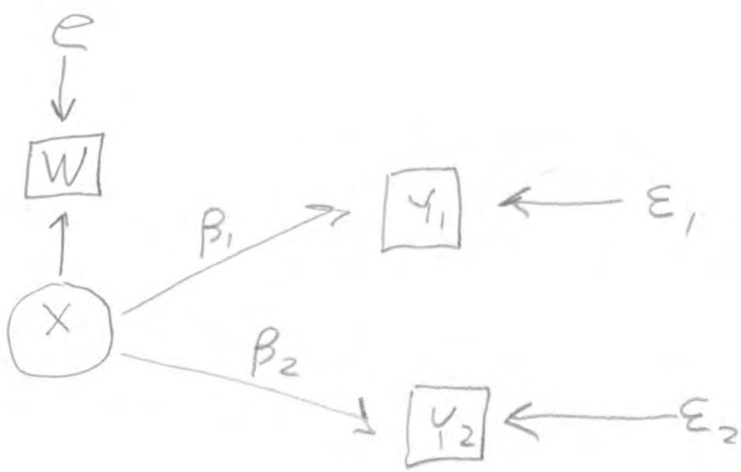
$$= cov(\beta x + \varepsilon + e_2, z)\, cov(x, z)^{-1}$$

$$= \beta \underbrace{cov(x, z)}_{K^T} \underbrace{cov(x, z)^{-1}}_{(K^T)^{-1}} = \beta$$ only fails if K^{-1} does not exist.

(h) Because if not identifiable, convergence in probability is impossible. The set of parameter values where the columns of K are linearly dependent is of lower dimension, so it's a set of volume zero.

5

(a)



(b) Unknown parameters are $(\beta_1, \beta_2, F_x, F_e, F_{\varepsilon_1}, F_{\varepsilon_2})$

(c) $\theta = (\beta_1, \beta_2, \phi, \omega, \psi_1, \psi_2)$

(d) Yes. There are $3(3+1)/2 = 6$ equations in 6 unknown parameters.

(e)

$$\Sigma = \begin{array}{c|ccc} & W & Y_1 & Y_2 \\ \hline W & \phi+\omega & \beta_1\phi & \beta_2\phi \\ Y_1 & & \beta_1^2\phi+\psi_1 & \beta_1\beta_2\phi \\ Y_2 & & & \beta_2^2\phi+\psi_2 \end{array}$$

$cov(W, Y_1) = cov(X+e, \beta_1 X + \varepsilon_1) = \beta_1\phi$

$cov(Y_1, Y_2) = cov(\beta_1 X + \varepsilon_1, \beta_2 X + \varepsilon_2) = \beta_1\beta_2\phi$

(f) β_1 is identifiable when $\beta_1 = 0$, because $\sigma_{12} = 0$ if and only if $\beta_1 = 0$

(5g) No, because if $\beta_1 = 0$, $\sigma_{12} = 0$ and $\sigma_{23} = 0$. That means ϕ can't be identified and we can't get ω from σ_{11} by subtraction.

(h) If $\beta_1 \neq 0$ and $\beta_2 = 0$,

$$\Sigma = \left(\begin{array}{c|c|c} \phi+\omega & \beta_1\phi & 0 \\ \hline & \beta_1^2\phi+\psi_1 & 0 \\ \hline & & \psi_2 \end{array} \right)$$

	β_1	β_2	ϕ	ω	ψ_1	ψ_2
θ_1	1/2	0	4	1	4	1
θ_2	2	0	1	4	1	1

Both yield

$$\Sigma = \left(\begin{array}{c|c|c} 5 & 2 & 0 \\ \hline & 5 & 0 \\ \hline & & 1 \end{array} \right)$$

There are infinitely many other correct answers

(i) $\beta_2 = 0$ iff $\sigma_{13} = 0$, so β_2 is identifiable if equal to zero.

(5j) If $\beta_1 \neq 0$ and $\beta_2 \neq 0$,

$$\phi = \sigma_{12}\sigma_{13}/\sigma_{23}$$

$$\beta_1 = \sigma_{12}/\phi$$

$$\beta_2 = \sigma_{13}/\phi$$

$$\omega = \sigma_{11} - \phi$$

$$\psi_1 = \sigma_{22} - \beta_1^2\phi$$

$$\psi_2 = \sigma_{33} - \beta_2^2\phi$$

(k) Since $\beta_1 = \sigma_{12}/\phi = \dfrac{\sigma_{12}}{\sigma_{12}\sigma_{13}/\sigma_{23}}$

$$= \frac{\cancel{\sigma_{12}}\sigma_{23}}{\cancel{\sigma_{12}}\sigma_{13}} = \frac{\sigma_{23}}{\sigma_{13}}, \text{ let } \hat{\beta}_1 = \frac{\hat{\sigma}_{23}}{\hat{\sigma}_{13}}$$

(l) Because β_1 is not identifiable at points where $\beta_2 = 0$, we can't get convergence in probability there, and consistency means convergence to the true parameter value at <u>every point</u> in the parameter space.

(5m) $\hat{\beta}_1 \xrightarrow{p} \beta_1$ for all points with $\beta_2 \neq 0$.

(n) Same number of equations as unknowns and parameters are identifiable, so the mapping $\Sigma = g(\theta)$ is 1-1. Invariance means $\hat{\theta} = g^{-1}(\hat{\Sigma})$. Note that even if $\beta_2 = 0$, $Pr(\hat{\beta}_2 = 0) = 0$, so with probability one, $\hat{\theta}$ lands in that part of the parameter space where θ is identifiable.

(o) If $\beta_1 = 0$, then β_1 is identifiable, but $\sigma_{12} = 0$ and $\sigma_{23} = 0$. This leaves 4 equations in the other 5 parameters, and the parameter count rule says they can't be identified except possibly on a set of volume zero in Θ. This means the restricted MLE $\hat{\theta}_0$ will not be unique, and estimation will fail.

(p) If $\beta_2 \neq 0$, then $\beta_1 = 0$ if and only if $\sigma_{12} = 0$ and $\sigma_{23} = 0$. This is unexpected because $H_0: \beta_1 = 0$ has only one = sign, and we expect a one df test.

(q) Because $\phi = \frac{\sigma_{12}\sigma_{13}}{\sigma_{23}}$ is a variance, $\frac{\sigma_{12}\sigma_{13}}{\sigma_{23}} > 0$

$\Longleftrightarrow \sigma_{12}\sigma_{13} > 0$ which is unexpected.

(6) (a) Parameter count rule: Now there are 6 equations in 7 unknowns.

(b) $\Sigma = cov\begin{pmatrix} W \\ Y_1 \\ Y_2 \end{pmatrix} =$

$$\begin{array}{c|c|c|c} & W & Y_1 & Y_2 \\ \hline W & \phi + \omega & \beta_1 \phi & \beta_2 \phi + c \\ \hline Y_1 & & \beta_1^2 \phi + \psi_1 & \beta_1(\beta_2 \phi + c) \\ \hline Y_2 & & & \beta_2^2 \phi + 2\beta_2 c + \psi_2 \end{array}$$

$W = X + e$

$Y_1 = \beta_1 X + \varepsilon_1$

$Y_2 = \beta_2 X + \varepsilon_2$

c

$$cov(Y_1, Y_2) = Cov(\beta_1 X + \varepsilon_1, \beta_2 X + \varepsilon_2)$$
$$= \beta_1 \beta_2 \phi + \beta_1 c$$

$$cov(Y_2) = cov(\beta_2 X + \varepsilon_2, \beta_2 X + \varepsilon_2)$$
$$= \beta_2^2 \phi + \beta_2 c + \beta_2 c + \psi_2$$

It's the same covariance matrix except for the third column.

(c) $\hat{\beta}_1 = \frac{\hat{\sigma}_{23}}{\hat{\sigma}_{13}}$, Same as (5d).

(d) Need $\beta_2 \neq 0$ or $c \neq 0$, or both.

(e) $H_0: \sigma_{13} = \sigma_{23} = 0$

(f) i. $H_0: \sigma_{12} = \sigma_{23} = 0$

ii. Yes. Test $H_0: \sigma_{12} = 0$

iii. Yes. Look at the sign of σ_{12}.