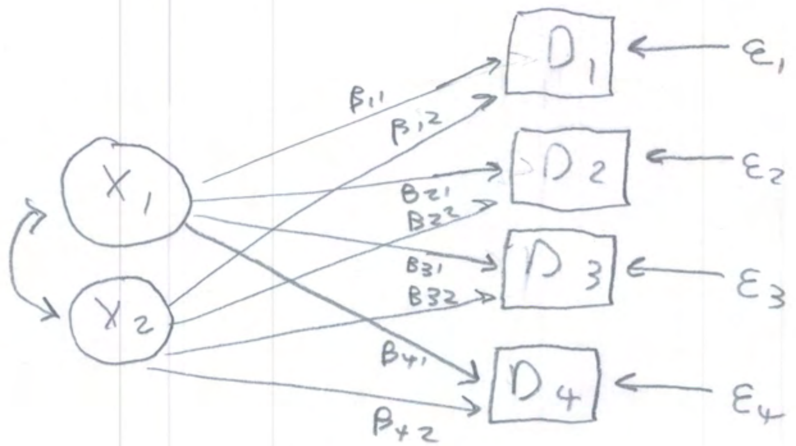
STA431S23 Assignment 2

(1)

(a)



(b) $\theta = (\mu_1\ \mu_2\ \sigma_{11}\ \sigma_{12}\ \sigma_{22}\ \alpha_1\ \alpha_2\ \alpha_3\ \alpha_4$
$\beta_{11}\ \beta_{12}\ \beta_{21}\ \beta_{22}\ \beta_{31}\ \beta_{32}\ \beta_{41}\ \beta_{42}$
$\sigma_1^2\ \sigma_2^2\ \sigma_3^2\ \sigma_4^2\ )$ That's 21, where

$\mu_1 = E(X_1)$, $\mu_2 = E(X_2)$, $\sigma_{11} = Var(X_1)$ $\sigma_{12} = cov(X_1, X_2)$
$\sigma_{22} = Var(X_2)$, $\sigma_j^2 = Var(\epsilon_j)$ for $j = 1, \ldots, 4$.

(2) Let $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$

Then $AB = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \neq BA = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$

(3) A and B must exist, because if not, $|A|=0$ or $|B|=0$ or both, and $|AB|=|A|\,|B|=0$
But $AB=I \Rightarrow |AB|=|I|=1$. This implies $|A|\neq 0$ and $|B|\neq 0$, and both inverses exist!

So $AB=I \Rightarrow A^{-1}AB=A^{-1}I \Rightarrow B=A^{-1}$
$AB=I \Rightarrow ABB^{-1}=IB^{-1} \Rightarrow A=B^{-1}$ ▫

(4) $AB=I$ and $AB=I$ so $AB=AC$

$$\Rightarrow \underbrace{BA}_{I}B=\underbrace{BA}_{I}C \Rightarrow B=C$$

(5) Inverse is only defined for square matrices.

(6) $$AB(B^{-1}A^{-1})=A\underbrace{BB^{-1}}_{I}A^{-1}=AA^{-1}=I$$

(7) $AB=I \Longrightarrow (AB)^T=I^T$
$\Longrightarrow B^TA^T=I^T=I$ done

(8)
$$a^Ta=\sum_{j=1}^{n}a_j^2 \geq 0$$

9 (a) $Ax = \lambda x \Rightarrow x^T A x = x^T \lambda x$

$= \lambda x^T x = \lambda$, so $0 < x^T A x = \lambda$

(b) $AA^{-1} = CD\underbrace{C^T C}D^{-1}C^T = C\underbrace{DD^{-1}}C^T$

$= CC^T = I$

(c) $A^{1/2}A^{\frac{1}{2}} = CD^{1/2}\underbrace{C^T C}D^{\frac{1}{2}}C^T$

$= CD^{\frac{1}{2}}D^{\frac{1}{2}}C^T = CDC^T = A$

(d) (i) $A^{-1/2}A^{-1/2} = CD^{-1/2}\underbrace{C^T C}D^{-1/2}C^T$

$= CD^{-1/2}D^{-1/2}C^T$

$= CD^{-1}C^T = A^{-1}$

(ii) $A^{-1/2}A^{1/2} = CD^{-1/2}\underbrace{C^T C}D^{1/2}C^T$

$= C\underbrace{D^{-\frac{1}{2}}D^{\frac{1}{2}}}C^T = CC^T = I$

(10) $tr(\Sigma) = tr(\underbrace{CD}_{A}\underbrace{C^T}_{B}) = tr(\underbrace{C^T C}_{I} D) = tr(D)$,

the sum of eigenvalues.

(11) $|\Sigma| = |CDC^T| = |C|\,|D|\,|C^T|$

$= |C|\,|C^T|\,|D| = |CC^T|\,|D|$

$= |I|\,|D| = |D|$ = product of eigenvalues

(12) Let v have a one in position j, and zeros elsewhere. Then

$$v^T A v = (0\,0 \cdots 1 \cdots 0)\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ a_{j1} & \cdots & a_{jn} \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \\ 0 \end{pmatrix}$$

$$= (a_{j1}\ a_{j2} \cdots a_{jn})\begin{pmatrix} 0 \\ \vdots \\ 1 \\ 0 \\ 0 \end{pmatrix} = a_{jj} > 0$$

because positive definite.

(13) Done in lecture. See Random Vectors and multivariate Normal, slide 9.

(14) $$Cov(Ax, By) = E\{(Ax - A\mu_x)(By - B\mu_y)^T\}$$
$$= E\{A(x-\mu_x)\left(B(y-\mu_y)\right)^T\}$$
$$= E\{A(x-\mu_x)(y-\mu_y)^T B^T\}$$
$$= A\,E\{(x-\mu_x)(y-\mu_y)^T\}B^T = A\,cov(x,y)\,B^T$$

(15) This is false, though some people can prove it anyway.

Let x_1 and x_2 be independent random variables, with $Var(x_1) = \sigma_1^2 > 0$ and $Var(x_2) = \sigma_2^2 > 0$. Then

$$cov\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{pmatrix} \neq \underset{\sim}{0}$$

For example, they could be standard normal.

(16) $\text{Cov}(x+c) = E\{(x+c-(\mu+c))(x+c-(\mu+c))^T\}$

$= E\{(x+\cancel{c}-\mu-\cancel{c})(x+\cancel{c}-\mu-\cancel{c})^T\}$

$= E\{(x-\mu)(x-\mu)^T\} = \text{cov}(x)$

(17) $\text{Var}(y) = v^T\Sigma v$, must be greater than or equal to zero since variances are non-negative.

(18) (a) By Problem 17, $v^T\Sigma v \geq 0$ for <u>any</u> $p\times 1$ vector v. Then if (λ, x) is an eigenvalue, eigenvector pair for Σ,

$$\Sigma x = \lambda x \implies x^T\Sigma x = x^T\lambda x = \lambda x^T x$$
$$= \lambda \geq 0$$

(b) Because the determinant is the product of the eigenvalues, which are non-negative by (a)

(18c) $$\begin{vmatrix} Var(x) & Cov(x,y) \\ Cov(x,y) & Var(y) \end{vmatrix} \geq 0$$

$$\iff Var(x)\,Var(y) - Cov(x,y)^2 \geq 0$$

$$\iff Cov(x,y)^2 \leq Var(x)\,Var(y)$$

$$\iff |Cov(x,y)| \leq \sqrt{Var(x)\,Var(y)}$$

$$\iff \left| \frac{Cov(x,y)}{Var(x)\,Var(y)} \right| \leq 1$$

$$\iff |Corr(x,y)| \leq 1$$

$$\iff -1 \leq Corr(x,y) \leq 1$$

$\blacksquare$

(19) (a) $\mathrm{Cov}(x_i, y_j)$

(b) $\mathrm{Cov}(x+y) = \mathrm{cov}(x+y, x+y) = \mathrm{Cov}(x,x)$
$+ \mathrm{Cov}(x,y) + \mathrm{Cov}(y,x) + \mathrm{cov}(y,y)$
$= \Sigma_x + \Sigma_{xy} + \Sigma_{xy}^T + \Sigma_y.$

(c) If $\mathrm{Cov}(x_i\, y_j) = 0$ for all $i \neq j$, $\Sigma_{xy} = 0$ and
$\mathrm{Cov}(x,y) = \Sigma_x + \Sigma_y$

(d) $\mathrm{Cov}(x+c, y+d)$
$= E\{(x+c-(\mu_x+c))(y+d-(\mu_y+d))^T\}$
$= E\{(x-\mu_x)(y-\mu_y)^T\} = \mathrm{Cov}(x,y) = \Sigma_{xy}$

(20) $\mathrm{Cov}(x_1+x_2, y_1+y_2) = \mathrm{Cov}(x_1, y_1) + \mathrm{Cov}(x_1, y_2)$
$+ \mathrm{Cov}(x_2\, y_1) + \mathrm{Cov}(x_2\, y_2)$

(21) This is false. Let x be 3×1 and y be 2×1.
Then $\mathrm{Cov}(x,y)$ is 3×2, and $\mathrm{cov}(y,x)$ is 2×3.

(22) False again. Let $x = z + e_1$ & $y = z + e_2$, with
e_1, e_2 & z independent & $\mathrm{cov}(z) = I$.
Then $\mathrm{cov}(x,y) = \mathrm{cov}(z+e_1, z+e_2) = \mathrm{Cov}(z,z) + 0$
$= \mathrm{cov}(z) = I \neq 0$.