University of Toronto at Mississauga

December Examinations 2004

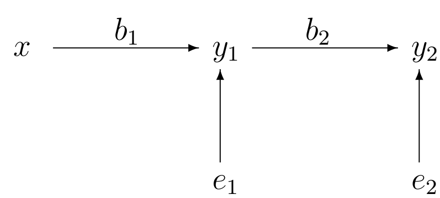
STA 313f

Duration - 3 hours

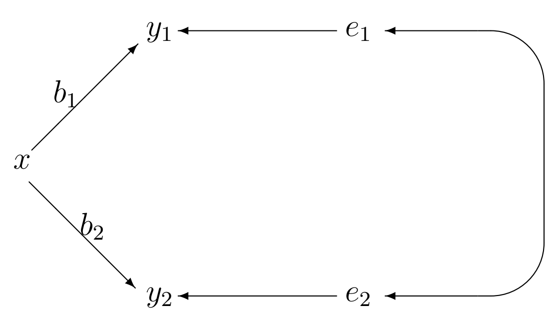
Aids allowed: None. Formula Sheet will be supplied.

For the structural equation models in this exam, all expected values are zero. The usual notation applies. That is, x and y variables are manifest, F variables are latent, and e variables are error terms. In path diagrams, unlabeled one-headed straight arrows correspond to a weight of one, while two-headed curved arrows (which are always unlabeled) represent non-zero covariances. Objects in **boldface** are **matrices** or **vectors**.

1. (4 Points) Is the following model identified? Answer Yes or No and say why. No marks for the right answer without a good reason.



2. (4 Points) Is the following model identified? Answer Yes or No and say why. No marks for the right answer without a good reason.



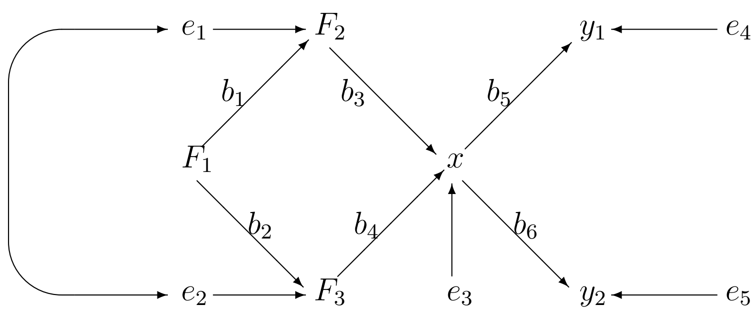
3. (4 Points) Is the following model identified? Answer Yes or No and say why. No marks for the right answer without a good reason.

$$
\begin{aligned}
y_1 &= b_1 x_1 + b_2 x_2 + e_1 \\
y_2 &= b_3 x_2 + b_4 x_3 + e_2 \\
y_3 &= b_5 y_1 + b_6 y_2 + e_3,
\end{aligned}
$$

where all random variables have expected value zero, the vector $\mathbf{x} = (x_1, x_2, x_3)'$ is multivariate normal with a covariance matrix that is *not* diagonal, and the error terms e_1, e_2 and e_3 are normal, and independent of $\mathbf{x}$ and of each other.

Continued on Page 2

4. (4 Points) Is the following path model identified? Answer Yes or No and say why. No marks for the right answer without a good reason.



5. (4 Points) Is the following model identified? Answer Yes or No and say why. No marks for the right answer without a good reason.

$$
\begin{aligned}
y_1 &= x + e_1 \\
y_2 &= x + y_1 + e_2 \\
y_3 &= x + y_1 + y_2 + e_3 \\
y_4 &= x + y_1 + y_2 + y_3 + e_4 \\
y_5 &= x + y_1 + y_2 + y_3 + y_4 + e_5,
\end{aligned}
$$

where $x \sim N(0, \sigma_x^2)$, $e_j \sim N(0, \sigma_j^2)$ for $j = 1, \ldots, 5$, and the e variables are all independent of each other and independent of x.

6. (4 Points) Is the following model identified? Answer Yes or No and say why. No marks for the right answer without a good reason.

$$
\begin{aligned}
y_1 &= \gamma_1 F + e_1 \\
y_2 &= \gamma_2 F + e_2,
\end{aligned}
$$

where $\gamma_1 > 0$. The error terms e_1 and e_2 are independent of F and independent of each other.

7. (4 Points) Is the following model identified? Answer Yes or No and say why. No marks for the right answer without a good reason.

$$
\begin{aligned}
y_1 &= b_1 x_1 + b_2 x_2 + e_1 \\
y_2 &= b_3 x_3 + b_4 x_4 + e_2,
\end{aligned}
$$

where $x_i \sim N(0, \sigma_i^2)$, $Cov(x_1, x_2) = \kappa_1$, $Cov(x_3, x_4) = \kappa_2$, x_1 and x_2 are independent of x_3 and x_4, $Cov(e_1, e_2) = \kappa_e$, and the error terms e_1 and e_2 are independent of all the x variables.

Continued on Page 3

8. (6 Points) Let $\mathbf{X}$ be a $p \times 1$ random vector, and let $\mathbf{Y}$ be a $q \times 1$ random vector In addition, let $\mathbf{X}$ and $\mathbf{Y}$ be independent. Show $E(\mathbf{X}\mathbf{Y}') = E(\mathbf{X})E(\mathbf{Y}')$.

9. (8 Points) Let $\mathbf{X}_1, \ldots, \mathbf{X}_n$ be independent and identically distributed multivariate normal random vectors, all with the same mean $\boldsymbol{\mu}$ and the same variance-covariance matrix $\boldsymbol{\Sigma}$. Setting $\boldsymbol{\mu} = \overline{\mathbf{x}}$ (this maximizes the likelihood over $\boldsymbol{\mu}$ for any $\boldsymbol{\Sigma}$, but you don't have to prove it), derive an expression for $-2 \log L(\overline{\mathbf{x}}, \boldsymbol{\Sigma})$ that depends on the sample data only through $\widehat{\boldsymbol{\Sigma}}$ You may use the expression for $\widehat{\boldsymbol{\Sigma}}$ from the formula sheet.

   Again, you are simplifying an expression for minus two times the log likelihood with $\boldsymbol{\mu}$ set to $\overline{\mathbf{x}}$, so that it is a function of $\boldsymbol{\Sigma}$ and the sample data. Your final answer must depend on the sample data *only* through $\widehat{\boldsymbol{\Sigma}}$. Show your work.

10. (8 Points) Let $X_1, \ldots, X_n$ be independent and identically distributed *univariate* normal random variables, all with the same mean μ and the same variance σ^2. Derive a large-sample likelihood ratio test for $H_0 : \sigma^2 = 1$. Show your work and *simplify*! If you know the MLEs of μ and σ^2, you do not have to derive them. Give a formula for G and *circle your final answer*. What are the degrees of freedom for this test?

11. (10 Points) Consider this model:

$$
\begin{aligned}
F &= bx + e_1 \\
y_1 &= F + e_2 \\
y_2 &= F + e_3,
\end{aligned}
$$

   where all random variables are normal with expected value zero, $V(x) = \sigma_x^2$, $V(e_j) = \sigma_j^2$, and the error terms are independent of x and of each other.

   (a) Draw a path diagram.

   (b) What is the parameter θ?

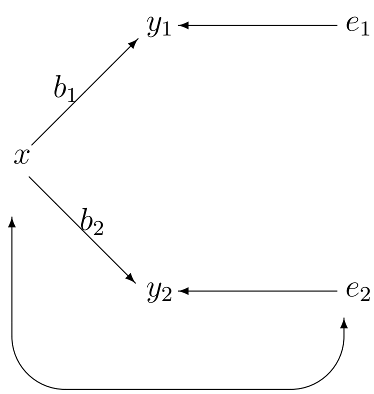
   (c) Is this model saturated? Answer Yes or No.

   (d) Give the variance-covariance matrix of the manifest variables in terms of the parameter θ; show your work. Each cell in your matrix should contain a formula for the variance or covariance in terms of quantities like σ_1^2, σ_2^2, and so on.

   (e) Is this model identified? Answer Yes or No and justify your answer.

Continued on Page 4

12. (8 Points) Is the following model identified? Answer Yes or No and say why. No marks for the right answer without a good reason.



13. (9 Points) Let $\mathbf{y} = \boldsymbol{\Gamma}\mathbf{x} + \boldsymbol{\zeta}$, with $V(\mathbf{x}) = \boldsymbol{\Phi}$, $V(\boldsymbol{\zeta}) = \boldsymbol{\Psi}$, and $\mathbf{x}$ and $\boldsymbol{\zeta}$ independent. Of course $\mathbf{x}$ and $\mathbf{y}$ are manifest. Is this model identified? Answer Yes or No, and prove it. If the model is identified, this time solve explicitly for the parameters; don't just say you can do it.

14. (9 Points) Let

$$\begin{aligned} \mathbf{x}_1 &= \boldsymbol{\xi} + \boldsymbol{\delta}_1 \\ \mathbf{x}_2 &= \boldsymbol{\xi} + \boldsymbol{\delta}_2 \\ \mathbf{y} &= \boldsymbol{\Gamma}\boldsymbol{\xi} + \boldsymbol{\zeta}, \end{aligned}$$

with $\boldsymbol{\delta}_1$, $\boldsymbol{\delta}_2$, $\boldsymbol{\zeta}$ and $\boldsymbol{\xi}$ all independent, $V(\boldsymbol{\xi}) = \boldsymbol{\Phi}$, $V(\boldsymbol{\zeta}) = \boldsymbol{\Psi}$, $V(\boldsymbol{\delta}_1) = \boldsymbol{\Theta}_1$, $V(\boldsymbol{\delta}_2) = \boldsymbol{\Theta}_2$, and $\boldsymbol{\Theta}_1 \neq \boldsymbol{\Theta}_2$. Is this model identified? Answer Yes or No, and prove it.

Note: If you were able to do Question 13, you will see that this one is quite similar. Therefore, *if* you did Question 13 correctly, you can just describe how you would try to solve for the model parameters here, without actually doing it. You do have to give a little detail; don't just say you can do it or that you can't do it.

15. (6 Points) In this question, you will partly connect the notation of the LISREL and EQS models.

    (a) Express the random vector $\boldsymbol{\xi}$ of the EQS model as a partitioned vector; its elements are random vectors from the LISREL model.

    (b) Express the covariance matrix Φ of the EQS model as a partitioned matrix; its nonzero elements are matrices from the LISREL model.

16. (8 Points) If you examine the LISREL model, you will see that it contains a factor analysis model for the latent exogenous variables, and another factor analysis model for the latent endogenous variables. Consider just the model for the exogenous variables – that is, the $\mathbf{x} = \boldsymbol{\Delta}_x\boldsymbol{\xi} + \boldsymbol{\delta}$ part. Is this model identified? Answer Yes or No, and prove your answer. Show details; don't just say you can solve for the parameters or that you cannot solve for them.

Total marks = 100 points