Structural Equation Models: The General Case[1]

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Features of Structural Equation Models

- Multiple equations.
- All the variables are random.
- An explanatory variable in one equation can be the response variable in another equation.
- Models are represented by path diagrams.
- Identifiability is always an issue.
- The statistical models are models of influence. They are often called *causal models*.

Correlation versus Causation

- The path diagrams deliberately imply influence. If $A \to B$, we are saying A *contributes* to B, or partly *causes* it.
- Data are usually observational. The correlation-causation issue does not go away.
- You may be able to argue on theoretical grounds that $A \to B$ is more believable than $B \to A$.
- If you have a causal model, you may be able to test whether it's compatible with the data.

Modest changes in notation

$$
\begin{aligned}
Y_{i,1} &= \alpha_1 + \gamma_1 X_{i,1} + \gamma_2 X_{i,2} + \epsilon_{i,1} \\
Y_{i,2} &= \alpha_2 + \beta Y_{i,1} + \epsilon_{i,2}
\end{aligned}
$$

- Regression coefficients (links between exogenous variables and endogenous variables) are now called gamma instead of beta.
- Betas are used for links between endogenous variables.
- Intercepts are alphas but they will soon disappear.

Losing the intercepts and expected values

- Mostly the intercepts and expected values are not identifiable anyway, as in multiple regression with measurement error.
- We have a chance to identify a *function* of the parameter vector – the parameters that appear in the covariance matrix $\mathbf{\Sigma}$ of an observable data vector. $\mathbf{\Sigma} = cov(\mathbf{D}_i)$.
- Denote the vector of parameters that appear in $\mathbf{\Sigma}$ by $\boldsymbol{\theta}$.
- Re-parameterize. The new parameter vector is $(\boldsymbol{\theta}, \boldsymbol{\mu})$, where $\boldsymbol{\mu} = E(\mathbf{D}_i)$.
- Estimate $\boldsymbol{\mu}$ with $\overline{\mathbf{D}}$, forget it, and concentrate on $\boldsymbol{\theta}$.
- To make calculation of the covariance matrix easier, write the model equations in centered form. The little letters c over the variables are invisible.
- From this point on the models *seemingly* have zero means and no intercepts.

A General Two-Stage Model

Stage 1 is the latent variable model and Stage 2 is the measurement model.

$$\begin{aligned} \mathbf{Y}_i &= \boldsymbol{\beta}\mathbf{Y}_i + \boldsymbol{\Gamma}\mathbf{X}_i + \boldsymbol{\epsilon}_i \\ \mathbf{F}_i &= \left(\begin{array}{c} \mathbf{X}_i \\ \mathbf{Y}_i \end{array} \right) \\ \mathbf{D}_i &= \boldsymbol{\Lambda}\mathbf{F}_i + \mathbf{e}_i \end{aligned}$$

- $\mathbf{D}_i$ (the data) are observable. All other variables are latent.
- $\mathbf{Y}_i = \boldsymbol{\beta}\mathbf{Y}_i + \boldsymbol{\Gamma}\mathbf{X}_i + \boldsymbol{\epsilon}_i$ is called the *Latent Variable Model*.
- The latent vectors $\mathbf{X}_i$ and $\mathbf{Y}_i$ are collected into a *factor* $\mathbf{F}_i$. This is *not* a categorical explanatory variable, the usual meaning of “factor” in experimental design.
- $\mathbf{D}_i = \boldsymbol{\Lambda}\mathbf{F}_i + \mathbf{e}_i$ is called the *Measurement Model*.

$\mathbf{Y}_i = \boldsymbol{\beta}\mathbf{Y}_i + \boldsymbol{\Gamma}\mathbf{X}_i + \boldsymbol{\epsilon}_i \quad \mathbf{F}_i = \begin{pmatrix} \mathbf{X}_i \\ \mathbf{Y}_i \end{pmatrix} \quad \mathbf{D}_i = \boldsymbol{\Lambda}\mathbf{F}_i + \mathbf{e}_i$

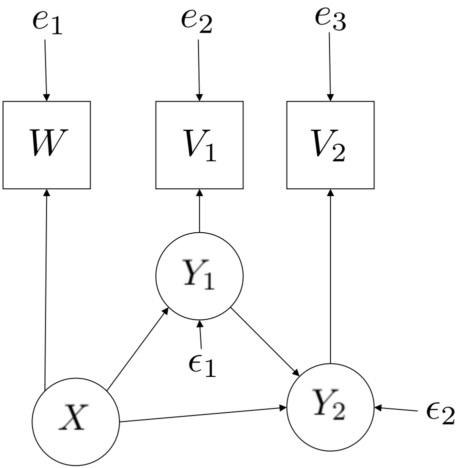
- $\mathbf{Y}_i$ is a $q \times 1$ random vector.
- $\boldsymbol{\beta}$ is a $q \times q$ matrix of constants with zeros on the main diagonal.
- $\mathbf{X}_i$ is a $p \times 1$ random vector.
- $\boldsymbol{\Gamma}$ is a $q \times p$ matrix of constants.
- $\boldsymbol{\epsilon}_i$ is a $q \times 1$ random vector.
- $\mathbf{F}_i$ (F for Factor) is just $\mathbf{X}_i$ stacked on top of $\mathbf{Y}_i$. It is a $(p+q) \times 1$ random vector.
- $\mathbf{D}_i$ is a $k \times 1$ random vector. Sometimes, $\mathbf{D}_i = \begin{pmatrix} \mathbf{W}_i \\ \mathbf{V}_i \end{pmatrix}$.
- $\boldsymbol{\Lambda}$ is a $k \times (p+q)$ matrix of constants: "factor loadings."
- $\mathbf{e}_i$ is a $k \times 1$ random vector.
- $\mathbf{X}_i$, $\boldsymbol{\epsilon}_i$ and $\mathbf{e}_i$ are independent.

Covariance matrices
More notation

$$\begin{aligned} \mathbf{Y}_i &= \boldsymbol{\beta}\mathbf{Y}_i + \boldsymbol{\Gamma}\mathbf{X}_i + \boldsymbol{\epsilon}_i \\ \mathbf{F}_i &= \left(\begin{array}{c} \mathbf{X}_i \\ \mathbf{Y}_i \end{array} \right) \\ \mathbf{D}_i &= \boldsymbol{\Lambda}\mathbf{F}_i + \mathbf{e}_i \end{aligned}$$

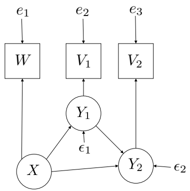
$$\begin{aligned} cov(\mathbf{X}_i) &= \boldsymbol{\Phi}_x \\ cov(\boldsymbol{\epsilon}_i) &= \boldsymbol{\Psi} \\ cov(\mathbf{F}_i) &= \boldsymbol{\Phi} = \left(\begin{array}{cc} cov(\mathbf{X}_i) & cov(\mathbf{X}_i, \mathbf{Y}_i) \\ cov(\mathbf{Y}_i, \mathbf{X}_i) & cov(\mathbf{Y}_i) \end{array} \right) = \left(\begin{array}{cc} \boldsymbol{\Phi}_{11} & \boldsymbol{\Phi}_{12} \\ \boldsymbol{\Phi}_{12}^\top & \boldsymbol{\Phi}_{22} \end{array} \right) \\ cov(\mathbf{e}_i) &= \boldsymbol{\Omega} \\ cov(\mathbf{D}_i) &= \boldsymbol{\Sigma} \end{aligned}$$

Example: A Path Model with Measurement Error



$$
\begin{aligned}
Y_{i,1} &= \gamma_1 X_i + \epsilon_{i,1} \\
Y_{i,2} &= \beta Y_{i,1} + \gamma_2 X_i + \epsilon_{i,2} \\
W_i &= X_i + e_{i,1} \\
V_{i,1} &= Y_{i,1} + e_{i,2} \\
V_{i,2} &= Y_{i,2} + e_{i,3}
\end{aligned}
$$

Matrix Form



$$
\begin{aligned}
Y_{i,1} &= \gamma_1 X_i + \epsilon_{i,1} \\
Y_{i,2} &= \beta Y_{i,1} + \gamma_2 X_i + \epsilon_{i,2} \\
W_i &= X_i + e_{i,1} \\
V_{i,1} &= Y_{i,1} + e_{i,2} \\
V_{i,2} &= Y_{i,2} + e_{i,3}
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{Y}_i &= \boldsymbol{\beta} \mathbf{Y}_i + \boldsymbol{\Gamma} \mathbf{X}_i + \boldsymbol{\epsilon}_i \\
\mathbf{F}_i &= \begin{pmatrix} \mathbf{X}_i \\ \mathbf{Y}_i \end{pmatrix} \\
\mathbf{D}_i &= \boldsymbol{\Lambda} \mathbf{F}_i + \mathbf{e}_i
\end{aligned}
$$

$$
\begin{array}{ccccccccc}
\mathbf{Y}_i & = & \boldsymbol{\beta} & \mathbf{Y}_i & + & \boldsymbol{\Gamma} & \mathbf{X}_i & + & \boldsymbol{\epsilon}_i \\
\begin{pmatrix} Y_{i,1} \\ Y_{i,2} \end{pmatrix} & = & \begin{pmatrix} 0 & 0 \\ \beta & 0 \end{pmatrix} & \begin{pmatrix} Y_{i,1} \\ Y_{i,2} \end{pmatrix} & + & \begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix} & X_i & + & \begin{pmatrix} \epsilon_{i,1} \\ \epsilon_{i,2} \end{pmatrix}
\end{array}
$$

$$
\begin{array}{ccccccc}
\mathbf{D}_i & = & \boldsymbol{\Lambda} & \mathbf{F}_i & + & \mathbf{e}_i \\
\begin{pmatrix} W_i \\ V_{i,1} \\ V_{i,2} \end{pmatrix} & = & \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} & \begin{pmatrix} X_i \\ Y_{i,1} \\ Y_{i,2} \end{pmatrix} & + & \begin{pmatrix} e_{i,1} \\ e_{i,2} \\ e_{i,3} \end{pmatrix}
\end{array}
$$

Observable variables in the "latent" variable model $\mathbf{Y}_i = \boldsymbol{\beta}\mathbf{Y}_i + \boldsymbol{\Gamma}\mathbf{X}_i + \boldsymbol{\epsilon}_i$

Fairly common

- These present no problem.
- Let $P(e_j = 0) = 1$, so $Var(e_j) = 0$.
- And $Cov(e_i, e_j) = 0$
- Because if $P(e_j = 0) = 1$,

$$\begin{aligned} Cov(e_i, e_j) &= E(e_i e_j) - E(e_i)E(e_j) \\ &= E(e_i \cdot 0) - E(e_i) \cdot 0 \\ &= 0 - 0 = 0 \end{aligned}$$

- In $\boldsymbol{\Omega} = cov(\mathbf{e}_i)$, column j (and row j) are all zeros.
- $\boldsymbol{\Omega}$ singular, no problem.

What should you be able to do?

- Given a path diagram, write the model equations and say which exogenous variables are correlated with each other.
- Given the model equations and information about which exogenous variables are correlated with each other, draw the path diagram.
- Given either piece of information, write the model in matrix form and say what all the matrices are.
- Calculate model covariance matrices.
- Check identifiability.

Recall the notation

$$
\begin{aligned}
\mathbf{Y}_i &= \boldsymbol{\beta}\mathbf{Y}_i + \boldsymbol{\Gamma}\mathbf{X}_i + \boldsymbol{\epsilon}_i \\
\mathbf{F}_i &= \begin{pmatrix} \mathbf{X}_i \\ \mathbf{Y}_i \end{pmatrix} \\
\mathbf{D}_i &= \boldsymbol{\Lambda}\mathbf{F}_i + \mathbf{e}_i
\end{aligned}
$$

$$
\begin{aligned}
cov(\mathbf{X}_i) &= \boldsymbol{\Phi}_x \\
cov(\boldsymbol{\epsilon}_i) &= \boldsymbol{\Psi} \\
cov(\mathbf{F}_i) &= \boldsymbol{\Phi} = \begin{pmatrix} cov(\mathbf{X}_i) & cov(\mathbf{X}_i, \mathbf{Y}_i) \\ cov(\mathbf{Y}_i, \mathbf{X}_i) & cov(\mathbf{Y}_i) \end{pmatrix} = \begin{pmatrix} \boldsymbol{\Phi}_{11} & \boldsymbol{\Phi}_{12} \\ \boldsymbol{\Phi}_{12}^\top & \boldsymbol{\Phi}_{22} \end{pmatrix} \\
cov(\mathbf{e}_i) &= \boldsymbol{\Omega} \\
cov(\mathbf{D}_i) &= \boldsymbol{\Sigma}
\end{aligned}
$$

Calculate a general expression for $\boldsymbol{\Sigma}(\boldsymbol{\theta})$.

For the latent variable model, calculate $\mathbf{\Phi} = cov(\mathbf{F}_i)$

Have $cov(\mathbf{X}_i) = \mathbf{\Phi}_x$, need $cov(\mathbf{Y}_i)$ and $cov(\mathbf{X}_i, \mathbf{Y}_i)$

$$
\begin{aligned}
& \mathbf{Y}_i = \boldsymbol{\beta}\mathbf{Y}_i + \mathbf{\Gamma}\mathbf{X}_i + \boldsymbol{\epsilon}_i \\
\Rightarrow \quad & \mathbf{Y}_i - \boldsymbol{\beta}\mathbf{Y}_i = \mathbf{\Gamma}\mathbf{X}_i + \boldsymbol{\epsilon}_i \\
\Rightarrow \quad & \mathbf{I}\mathbf{Y}_i - \boldsymbol{\beta}\mathbf{Y}_i = \mathbf{\Gamma}\mathbf{X} + \boldsymbol{\epsilon}_i \\
\Rightarrow \quad & (\mathbf{I} - \boldsymbol{\beta})\mathbf{Y} = \mathbf{\Gamma}\mathbf{X}_i + \boldsymbol{\epsilon}_i \\
\Rightarrow \quad & (\mathbf{I} - \boldsymbol{\beta})^{-1}(\mathbf{I} - \boldsymbol{\beta})\mathbf{Y}_i = (\mathbf{I} - \boldsymbol{\beta})^{-1}(\mathbf{\Gamma}\mathbf{X}_i + \boldsymbol{\epsilon}_i) \\
\Rightarrow \quad & \mathbf{Y}_i = (\mathbf{I} - \boldsymbol{\beta})^{-1}(\mathbf{\Gamma}\mathbf{X}_i + \boldsymbol{\epsilon}_i)
\end{aligned}
$$

So,

$$
\begin{aligned}
cov(\mathbf{Y}_i) &= (\mathbf{I} - \boldsymbol{\beta})^{-1} cov(\mathbf{\Gamma}\mathbf{X}_i + \boldsymbol{\epsilon}_i)(\mathbf{I} - \boldsymbol{\beta})^{-1\top} \\
&= (\mathbf{I} - \boldsymbol{\beta})^{-1} \left(cov(\mathbf{\Gamma}\mathbf{X}_i) + cov(\boldsymbol{\epsilon}_i)\right) (\mathbf{I} - \boldsymbol{\beta}^\top)^{-1} \\
&= (\mathbf{I} - \boldsymbol{\beta})^{-1} \left(\mathbf{\Gamma}\mathbf{\Phi}_{11}\mathbf{\Gamma}^\top + \mathbf{\Psi}\right) (\mathbf{I} - \boldsymbol{\beta}^\top)^{-1}
\end{aligned}
$$

For the measurement model, calculate $\mathbf{\Sigma} = cov(\mathbf{D}_i)$

$$
\begin{aligned}
\mathbf{D}_i &= \mathbf{\Lambda}\mathbf{F}_i + \mathbf{e}_i \\
\Rightarrow cov(\mathbf{D}_i) &= cov(\mathbf{\Lambda}\mathbf{F}_i + \mathbf{e}_i) \\
&= cov(\mathbf{\Lambda}\mathbf{F}_i) + cov(\mathbf{e}_i) \\
&= \mathbf{\Lambda} cov(\mathbf{F}_i)\mathbf{\Lambda}^\top + cov(\mathbf{e}_i) \\
&= \mathbf{\Lambda}\mathbf{\Phi}\mathbf{\Lambda}^\top + \mathbf{\Omega} \\
&= \mathbf{\Sigma}
\end{aligned}
$$

Two-stage Proofs of Identifiability

Stage 1 is the latent variable model and Stage 2 is the measurement model.

- Show the parameters of the latent variable model $(\boldsymbol{\beta}, \boldsymbol{\Gamma}, \boldsymbol{\Phi}_{11}, \boldsymbol{\Psi})$ can be recovered from $\boldsymbol{\Phi} = cov(\mathbf{F}_i)$.
- Show the parameters of the measurement model $(\boldsymbol{\Lambda}, \boldsymbol{\Phi}, \boldsymbol{\Omega})$ can be recovered from $\boldsymbol{\Sigma} = cov(\mathbf{D}_i)$.
- This means all the parameters can be recovered from $\boldsymbol{\Sigma}$.
- Break a big problem into two smaller ones.
- Develop *rules* for checking identifiability at each stage.
- Just look at the path diagram.

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