STA 431s17 Assignment Three[1]

This assignment is on material from Lecture Slide Set 5: Statistical models and estimation. There is also some background in Section A.6 of Appendix A, but that section is dominated by numerical maximum likelihood with R, which is nice but not needed for this course.

1. Let $X_1, \ldots, X_n$ be a random sample from a Poisson distribution with expected value $\lambda > 0$.
   (a) What is the parameter of this model?
   (b) What is the parameter space? See the lecture slides for how to write it.
2. Let $Y_1, \ldots, Y_n$ be a random sample from a normal distribution with expected value μ and variance σ^2.
   (a) What are the parameters of this model?
   (b) What is the parameter space?
3. Independently for $i = 1, \ldots, n$, let $Y_i = \beta X_i + \epsilon_i$, where $X_i \sim N(\mu_x, \sigma_x^2)$, $\epsilon_i \sim N(0, \sigma_\epsilon^2)$, and X_i and ϵ_i are independent.
   (a) What are the parameters of this model?
   (b) What is the parameter space?
4. For $i = 1, \ldots, n$, let $Y_i = \beta_0 + \beta_1 x_{i1} + \cdots + \beta_k x_{ik} + \epsilon_i$, where

   $\beta_0, \ldots, \beta_k$ are unknown constants.

   x_{ij} are known constants.

   $\epsilon_1, \ldots, \epsilon_n$ are independent $N(0, \sigma^2)$ random variables.

   σ^2 is an unknown constant.

   $Y_1, \ldots, Y_n$ are observable random variables.

   (a) What are the parameters of this model?
   (b) What is the parameter space?
5. Let $X_1, \ldots, X_n$ be a random sample from a normal distribution with expected value μ and variance σ^2.
   (a) What is the parameter space for this model?
   (b) Derive the Maximum Likelihood Estimator of the pair $\theta = (\mu, \sigma^2)$. Show your work.
   (c) Find a Method of Moments estimator of θ. Use the fact that $E(X_i) = \mu$ and $Var(X_i) = \sigma^2$. This is very quick. Don't waste time and effort doing unnecessary things.
   (d) In the following R output, data are in the vector x. Based on this, give $\widehat{\theta}$. Your answer is a pair of numbers. I needed a calculator because R's `var` function uses $n-1$ in the denominator.

   ```
   > c(length(x),mean(x),var(x))
   [1]  20.0000  94.3800 155.1554
   ```

   (e) Give the maximum likelihood estimator of the standard deviation σ. The answer is a number. Do it the easy way. How do you know that this is okay?

[1] This assignment was prepared by Jerry Brunner, Department of Statistical Sciences, University of Toronto. The LaTeX source code is available from the course website: http://www.utstat.toronto.edu/~brunner/oldclass/431s17

6. The formula sheet has a useful expression for the multivariate normal likelihood.

   (a) Show that you understand the notation by giving the univariate version, in which $X_1, \ldots, X_n \stackrel{i.i.d}{\sim} N(\mu, \sigma^2)$. Your answer will have no matrix notation for the trace, transpose or inverse.

   (b) Now starting with the univariate normal density (also on the formula sheet), show that the univariate normal likelihood is the same as your answer to the previous question. Hint: Add and subtract $\overline{X}$.

   (c) How does this expression allow you to see *without differentiating* that the MLE of μ is $\overline{X}$?

7. Starting with the multivariate normal density on the formula sheet, derive the multivariate normal likelihood, also on the formula sheet. You will use $tr(\mathbf{AB}) = tr(\mathbf{AB})$ and other properties of the trace.

8. Independently for $i = 1, \ldots, n$, let $\mathbf{Y}_i = \boldsymbol{\beta}_0 + \boldsymbol{\beta}_1 \mathbf{X}_i + \boldsymbol{\epsilon}_i$, where

   - $\mathbf{Y}_i$ is an $q \times 1$ random vector of observable response variables; there are q response variables.
   - $\mathbf{X}_i$ is a $p \times 1$ observable random vector; there are p explanatory variables. $E(\mathbf{X}_i) = \boldsymbol{\mu}_x$ and $V(\mathbf{X}_i) = \boldsymbol{\Phi}_{p \times p}$. The positive definite matrix $\boldsymbol{\Phi}$ is unknown.
   - $\boldsymbol{\beta}_0$ is a $q \times 1$ matrix of unknown constants.
   - $\boldsymbol{\beta}_1$ is a $q \times p$ matrix of unknown constants.
   - $\boldsymbol{\epsilon}_i$ is a $q \times 1$ random vector with expected value zero and unknown positive definite variance-covariance matrix $V(\boldsymbol{\epsilon}_i) = \boldsymbol{\Psi}_{q \times q}$.
   - $\boldsymbol{\epsilon}_i$ is independent of $\mathbf{X}_i$.

   Letting $\mathbf{D}_i = \left(\begin{array}{c} \mathbf{X}_i \\ \hline \mathbf{Y}_i \end{array} \right)$, we have $V(\mathbf{D}_i) = \boldsymbol{\Sigma} = \left(\begin{array}{c|c} \boldsymbol{\Sigma}_x & \boldsymbol{\Sigma}_{xy} \\ \hline \boldsymbol{\Sigma}_{yx} & \boldsymbol{\Sigma}_y \end{array} \right)$, and $\widehat{\boldsymbol{\Sigma}} = \left(\begin{array}{c|c} \widehat{\boldsymbol{\Sigma}}_x & \widehat{\boldsymbol{\Sigma}}_{xy} \\ \hline \widehat{\boldsymbol{\Sigma}}_{yx} & \widehat{\boldsymbol{\Sigma}}_y \end{array} \right)$.

   (a) Start by writing $\boldsymbol{\Sigma}$ in terms of the unknown parameter matrices.

   (b) Give a Method of Moments Estimator for $\boldsymbol{\Phi}$. Just write it down.

   (c) Obtain formulas for the Method of Moments Estimators of $\boldsymbol{\beta}_1$, $\boldsymbol{\beta}_0$ and $\boldsymbol{\Psi}$. Show your work. You may give $\widehat{\boldsymbol{\beta}}_0$ in terms of $\widehat{\boldsymbol{\beta}}_1$, but simplify $\widehat{\boldsymbol{\Psi}}$.

   (d) If the distributions of $\mathbf{X}_i$ and $\boldsymbol{\epsilon}_i$ are multivariate normal, how do you know that your Method of Moments estimates are also the MLEs?

9. Let $X_1, \ldots, X_n$ be a random sample from a continuous distribution with density

$$f(x; \theta) = \frac{1}{\theta^{1/2}\sqrt{2\pi}} \, e^{-\frac{x^2}{2\theta}},$$

   where the parameter $\theta > 0$. Propose a reasonable estimator for the parameter θ, and use the Law of Large Numbers to show that your estimator is consistent.

10. Let $X_1, \ldots, X_n$ be a random sample from a Gamma distribution with $\alpha = \beta = \theta > 0$. That is, the density is

$$f(x; \theta) = \frac{1}{\theta^\theta \Gamma(\theta)} e^{-x/\theta} x^{\theta - 1},$$

   for $x > 0$. Let $\widehat{\theta} = \overline{X}_n$. Is $\widehat{\theta}$ consistent for θ? Answer Yes or No and prove your answer. Hint: The expected value of a Gamma random variable is $\alpha\beta$.

11. Independently for $i = 1, \ldots, n$, let

$$Y_i = \beta X_i + \epsilon_i,$$

where $E(X_i) = \mu_x$, $Var(X_i) = \sigma^2_x$, $E(\epsilon_i) = 0$, $Var(\epsilon_i) = \sigma^2_\epsilon$, and ϵ_i is independent of X_i. Let

$$\widehat{\beta} = \frac{\sum_{i=1}^n X_i Y_i}{\sum_{i=1}^n X_i^2}.$$

Is $\widehat{\beta}$ consistent for β? Answer Yes or No and prove your answer.

12. Another Method of Moments estimator for Problem 11 is $\widehat{\beta}_2 = \frac{\overline{Y}_n}{\overline{X}_n}$.

(a) Show that $\widehat{\beta}_2 \stackrel{p}{\to} \beta$ in most of the parameter space.

(b) However, consistency means that the estimator converges to the parameter in probability *everywhere* in the parameter space. Where does $\widehat{\beta}_2$ fail, and why?

13. Let $X_1, \ldots, X_n$ be a random sample from a distribution with expected value μ and variance σ^2_x. Independently of $X_1, \ldots, X_n$, let $Y_1, \ldots, Y_n$ be a random sample from a distribution with the same expected value μ and variance σ^2_y. Let Let $T_n = \alpha \overline{X}_n + (1-\alpha)\overline{Y}_n$, where $0 \leq \alpha \leq 1$. Is T_n always a consistent estimator of μ? Answer Yes or No and show your work.

14. Let $X_1, \ldots, X_n$ be a random sample from a distribution with mean μ. Show that $T_n = \frac{1}{n+400} \sum_{i=1}^n X_i$ is consistent for μ. Hint: If a sequence of constants $a_n \to a$ as an ordinary limit, you can view the constants as degenerate random variables and write $a_n \stackrel{p}{\to} a$. Then you can use continuous mapping and so on with confidence.

15. Let $X_1, \ldots, X_n$ be a random sample from a distribution with mean μ and variance σ^2. Prove that the sample variance $S^2 = \frac{\sum_{i=1}^n (X_i - \overline{X})^2}{n-1}$ is consistent for σ^2.

16. Let $(X_1, Y_1), \ldots, (X_n, Y_n)$ be a random sample from a bivariate distribution with $E(X_i) = \mu_x$, $E(Y_i) = \mu_y$, $Var(X_i) = \sigma^2_x$, $Var(Y_i) = \sigma^2_y$, and $Cov(X_i, Y_i) = \sigma_{xy}$. Show that the sample covariance $S_{xy} = \frac{\sum_{i=1}^n (X_i - \overline{X})(Y_i - \overline{Y})}{n-1}$ is a consistent estimator of σ_{xy}.

17. Let $X_1, \ldots, X_n$ be a random sample from a Poisson distribution with parameter λ. You know that $E(X_i) = Var(X_i) = \lambda$; there is no need to prove it.

From the Law of Large Numbers, it follows immediately that $\overline{X}_n$ is consistent for λ. Let

$$\widehat{\lambda} = \frac{\sum_{i=1}^n (X_i - \overline{X}_n)^2}{n-4}.$$

Is $\widehat{\lambda}$ also consistent for λ? Answer Yes or No and prove your answer.