More Linear Algebra[1]

STA431 Winter/Spring 2013

[1] See last slide for copyright information.

Overview

You already know about

- Matrices $\mathbf{A} = [a_{ij}]$
- Matrix addition and subtraction $\mathbf{A} + \mathbf{B} = [a_{ij} + b_{ij}]$
- Scalar multiplication $a\mathbf{B} = [a\, b_{ij}]$
- Matrix multiplication $\mathbf{AB} = \left[\sum_k a_{ik} b_{kj}\right]$
- Inverse $\mathbf{A}^{-1}\mathbf{A} = \mathbf{A}\mathbf{A}^{-1} = \mathbf{I}$
- Transpose $\mathbf{A}' = [a_{ji}]$
- Symmetric matrices $\mathbf{A} = \mathbf{A}'$
- Determinants
- Linear independence

Linear independence

$\mathbf{X}$ be an $n \times p$ matrix of constants.The columns of $\mathbf{X}$ are said to be *linearly dependent* if there exists $\mathbf{v} \neq \mathbf{0}$ with $\mathbf{X}\mathbf{v} = \mathbf{0}$. We will say that the columns of $\mathbf{X}$ are linearly *independent* if $\mathbf{X}\mathbf{v} = \mathbf{0}$ implies $\mathbf{v} = \mathbf{0}$.

For example, show that $\mathbf{A}^{-1}$ exists implies that the columns of $\mathbf{A}$ are linearly independent.

$$\mathbf{A}\mathbf{v} = \mathbf{0} \Rightarrow \mathbf{A}^{-1}\mathbf{A}\mathbf{v} = \mathbf{A}^{-1}\mathbf{0} \Rightarrow \mathbf{v} = \mathbf{0}$$

How to show $\mathbf{A}^{-1\prime} = \mathbf{A}^{\prime -1}$

Suppose $\mathbf{B} = \mathbf{A}^{-1}$, meaning $\mathbf{AB} = \mathbf{BA} = \mathbf{I}$. Must show two things: $\mathbf{B}'\mathbf{A}' = \mathbf{I}$ and $\mathbf{A}'\mathbf{B}' = \mathbf{I}$.

$$\begin{aligned} \mathbf{AB} = \mathbf{I} &\Rightarrow \mathbf{B}'\mathbf{A}' = \mathbf{I}' = \mathbf{I} \\ \mathbf{BA} = \mathbf{I} &\Rightarrow \mathbf{A}'\mathbf{B}' = \mathbf{I}' = \mathbf{I} \end{aligned}$$

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Trace of a square matrix: Sum of the diagonal elements

$$tr(\mathbf{A}) = \sum_{i=1}^{n} a_{i,i}.$$

- Of course $tr(\mathbf{A}+\mathbf{B}) = tr(\mathbf{A}) + tr(\mathbf{B})$, etc.
- But less obviously, even though $\mathbf{AB} \neq \mathbf{BA}$,
- $tr(\mathbf{AB}) = tr(\mathbf{BA})$

$tr(\mathbf{AB}) = tr(\mathbf{BA})$

Let $\mathbf{A}$ be an $r \times p$ matrix and $\mathbf{B}$ be a $p \times r$ matrix, so that the product matrices $\mathbf{AB}$ and $\mathbf{BA}$ are both defined.

$$
\begin{aligned}
tr(\mathbf{AB}) &= tr\left(\left[\sum_{k=1}^{p} a_{i,k} b_{k,j}\right]\right) \\
&= \sum_{i=1}^{r} \sum_{k=1}^{p} a_{i,k} b_{k,i} \\
&= \sum_{k=1}^{p} \sum_{i=1}^{r} b_{k,i} a_{i,k} \\
&= \sum_{i=1}^{p} \sum_{k=1}^{r} b_{i,k} a_{k,i} \qquad \text{(Switching } i \text{ and } k\text{)} \\
&= tr\left(\left[\sum_{k=1}^{r} b_{i,k} a_{k,j}\right]\right) \\
&= tr(\mathbf{BA})
\end{aligned}
$$

Eigenvalues and eigenvectors

Let $\mathbf{A} = [a_{i,j}]$ be an $n \times n$ matrix, so that the following applies to square matrices. $\mathbf{A}$ is said to have an *eigenvalue* λ and (non-zero) *eigenvector* $\mathbf{x}$ corresponding to λ if

$$\mathbf{A}\mathbf{x} = \lambda\mathbf{x}.$$

- Eigenvalues are the λ values that solve the determinantal equation $|\mathbf{A} - \lambda\mathbf{I}| = 0$.
- The determinant is the product of the eigenvalues: $|\mathbf{A}| = \prod_{i=1}^{n} \lambda_i$

Spectral decomposition of symmetric matrices

The *Spectral decomposition theorem* says that every square and symmetric matrix $\mathbf{A} = [a_{i,j}]$ may be written

$$\mathbf{A} = \mathbf{P}\boldsymbol{\Lambda}\mathbf{P}',$$

where the columns of $\mathbf{P}$ (which may also be denoted $\mathbf{x}_1, \ldots, \mathbf{x}_n$) are the eigenvectors of $\mathbf{A}$, and the diagonal matrix $\boldsymbol{\Lambda}$ contains the corresponding eigenvalues.

$$\boldsymbol{\Lambda} = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$

Because the eigenvectors are orthonormal, $\mathbf{P}$ is an orthogonal matrix; that is, $\mathbf{PP}' = \mathbf{P}'\mathbf{P} = \mathbf{I}$.

Positive definite matrices

The $n \times n$ matrix $\mathbf{A}$ is said to be *positive definite* if

$$\mathbf{y}'\mathbf{A}\mathbf{y} > 0$$

for *all* $n \times 1$ vectors $\mathbf{y} \neq \mathbf{0}$. It is called *non-negative definite* (or sometimes positive semi-definite) if $\mathbf{y}'\mathbf{A}\mathbf{y} \geq 0$.

Example: Show $\mathbf{X}'\mathbf{X}$ non-negative definite

Let $\mathbf{X}$ be an $n \times p$ matrix of real constants and $\mathbf{y}$ be $p \times 1$. Then $\mathbf{Z} = \mathbf{X}\mathbf{y}$ is $n \times 1$, and

$$
\begin{aligned}
& \mathbf{y}'\left(\mathbf{X}'\mathbf{X}\right)\mathbf{y} \\
= \; & (\mathbf{X}\mathbf{y})'(\mathbf{X}\mathbf{y}) \\
= \; & \mathbf{Z}'\mathbf{Z} \\
= \; & \sum_{i=1}^{n} Z_i^2 \geq 0
\end{aligned}
$$

Some properties of symmetric positive definite matrices

Variance-covariance matrices are often assumed positive definite.

Positive definite
$\Rightarrow$ All eigenvalues positive $\Leftrightarrow$ Determinant positive
$\Rightarrow$ Inverse exists $\Leftrightarrow$ Columns (rows) linearly independent

If a real symmetric matrix is also non-negative definite, as a variance-covariance matrix *must* be, Inverse exists $\Rightarrow$ Positive definite

Showing Positive definite $\Rightarrow$ Eigenvalues positive

For example

Let $\mathbf{A}$ be square and symmetric as well as positive definite.

- Spectral decomposition says $\mathbf{A} = \mathbf{P}\mathbf{\Lambda}\mathbf{P}'$.
- Using $\mathbf{y}'\mathbf{A}\mathbf{y} > 0$, let $\mathbf{y}$ be an eigenvector, say the third one.
- Because eigenvectors are orthonormal,

$$
\begin{aligned}
\mathbf{y}'\mathbf{A}\mathbf{y} &= \mathbf{y}'\mathbf{P}\mathbf{\Lambda}\mathbf{P}'\mathbf{y} \\
&= \begin{pmatrix} 0 & 0 & 1 & \cdots & 0 \end{pmatrix} \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix} \\
&= \lambda_3 \\
&> 0
\end{aligned}
$$

Square root matrices

Define

$$\mathbf{\Lambda}^{1/2} = \begin{pmatrix} \sqrt{\lambda_1} & 0 & \cdots & 0 \\ 0 & \sqrt{\lambda_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sqrt{\lambda_n} \end{pmatrix}$$

So that

$$\begin{aligned} \mathbf{\Lambda}^{1/2}\mathbf{\Lambda}^{1/2} &= \begin{pmatrix} \sqrt{\lambda_1} & 0 & \cdots & 0 \\ 0 & \sqrt{\lambda_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sqrt{\lambda_n} \end{pmatrix} \begin{pmatrix} \sqrt{\lambda_1} & 0 & \cdots & 0 \\ 0 & \sqrt{\lambda_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sqrt{\lambda_n} \end{pmatrix} \\ &= \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix} = \mathbf{\Lambda} \end{aligned}$$

For a general symmetric matrix $\mathbf{A}$

Define

$$\mathbf{A}^{1/2} = \mathbf{P}\boldsymbol{\Lambda}^{1/2}\mathbf{P}'$$

So that

$$\begin{aligned}
\mathbf{A}^{1/2}\mathbf{A}^{1/2} &= \mathbf{P}\boldsymbol{\Lambda}^{1/2}\mathbf{P}'\mathbf{P}\boldsymbol{\Lambda}^{1/2}\mathbf{P}' \\
&= \mathbf{P}\boldsymbol{\Lambda}^{1/2}\,\mathbf{I}\,\boldsymbol{\Lambda}^{1/2}\mathbf{P}' \\
&= \mathbf{P}\boldsymbol{\Lambda}^{1/2}\boldsymbol{\Lambda}^{1/2}\mathbf{P}' \\
&= \mathbf{P}\boldsymbol{\Lambda}\mathbf{P}' \\
&= \mathbf{A}
\end{aligned}$$

More about symmetric positive definite matrices

Show as exercises

Let $\mathbf{A}$ be symmetric and positive definite. Then $\mathbf{A}^{-1} = \mathbf{P}\boldsymbol{\Lambda}^{-1}\mathbf{P}'$.

Letting $\mathbf{B} = \mathbf{P}\boldsymbol{\Lambda}^{-1/2}\mathbf{P}'$,

$$\begin{aligned} \mathbf{B} &= \left(\mathbf{A}^{-1}\right)^{1/2} \\ \mathbf{B} &= \left(\mathbf{A}^{1/2}\right)^{-1} \end{aligned}$$

This justifies saying $\mathbf{A}^{-1/2} = \mathbf{P}\boldsymbol{\Lambda}^{-1/2}\mathbf{P}'$

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