Assessing How Well the Model Fits the Data

- Many models place *restrictions* on the moments (covariances). If the model is true then $\boldsymbol{\Sigma}$ can't be just any symmetric positive definite matrix.
- These restrictions can be stated as a null hypothesis (or set of null hypotheses). If H_0 is rejected, we conclude that the model is incorrect.

Recall Double Measurement

$$
\begin{aligned}
Y_i &= \beta_0 + \beta_1 X_i + \epsilon_i \\
W_{i,1} &= \nu_1 + X_i + e_{i,1} \\
W_{i,2} &= \nu_2 + X_i + e_{i,2},
\end{aligned}
$$

$$
\boldsymbol{\mu} = \begin{pmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \end{pmatrix} = \begin{pmatrix} \mu_x + \nu_1 \\ \mu_x + \nu_2 \\ \beta_0 + \beta_1 \mu_x \end{pmatrix}
$$

No restrictions on $\boldsymbol{\mu}$

$$\boldsymbol{\Sigma} = \left[\begin{array}{ccc} \phi + \omega_1 & \phi & \beta_1 \phi \\ & \phi + \omega_2 & \beta_1 \phi \\ & & \beta_1^2 \phi + \psi \end{array} \right]$$

- Six equations in five unknowns
- One *equality constraint*: $\sigma_{13}=\sigma_{23}$
- In general, if the parameter is identifiable, the number of equality constraints equals the number of equations minus the number of unknown parameters.
- One *inequality constraint*: $\sigma_{12} > 0$

Likelihood ratio test for testing the equality restrictions

- Null hypothesis: *r* equality constraints on **Σ** are true.
- Alternative hypothesis: **Σ** is unrestricted.
- *df=r*
- Say $\boldsymbol{\Sigma} \in \mathcal{M}$ (the moment space).
- $H_0 : \boldsymbol{\Sigma} \in \mathcal{M}_0 \; v.s. \; H_1 : \boldsymbol{\Sigma} \in \mathcal{M}$

$$H_0 : \boldsymbol{\Sigma} \in \mathcal{M}_0 \;\; v.s. \;\; H_1 : \boldsymbol{\Sigma} \in \mathcal{M}$$

$$\begin{aligned} G &= -2\ln\left(\frac{\max_{\boldsymbol{\Sigma}\in\mathcal{M}_0} L(\boldsymbol{\Sigma})}{\max_{\boldsymbol{\Sigma}\in\mathcal{M}} L(\boldsymbol{\Sigma})}\right) \\ &= -2\ln L(\boldsymbol{\Sigma}(\widehat{\boldsymbol{\theta}})) - [-2\ln L(\widehat{\boldsymbol{\Sigma}})] \end{aligned}$$

Maximize the likelihood over Θ, find MLE, compute Σ(MLE)
It will obey the equality constraints.

Simplify G: Recall likelihood of MVN

$$L(\boldsymbol{\mu}, \boldsymbol{\Sigma}) = |\boldsymbol{\Sigma}|^{-n/2}(2\pi)^{-nk/2} \exp -\frac{n}{2}\left\{tr(\widehat{\boldsymbol{\Sigma}}\boldsymbol{\Sigma}^{-1}) + (\overline{\mathbf{x}} - \boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\overline{\mathbf{x}} - \boldsymbol{\mu})\right\}$$

Estimate mu with x-bar and it's gone.

$$\begin{aligned} G &= -2\ln L(\boldsymbol{\Sigma}(\widehat{\boldsymbol{\theta}})) - [-2\ln L(\widehat{\boldsymbol{\Sigma}})] \\ &= n\left(tr(\widehat{\boldsymbol{\Sigma}}\boldsymbol{\Sigma}(\widehat{\boldsymbol{\theta}})^{-1}) + \ln|\boldsymbol{\Sigma}(\widehat{\boldsymbol{\theta}})| - \ln|\widehat{\boldsymbol{\Sigma}}| - k\right) \end{aligned}$$

A cute way to maximize the likelihood over θ in Θ

- Minimize G(θ) – just -2 ln(L(θ)) plus a constant

$$
\begin{aligned}
G(\boldsymbol{\theta}) &= -2\ln L(\boldsymbol{\Sigma}(\boldsymbol{\theta})) - [-2\ln L(\widehat{\boldsymbol{\Sigma}})] \\
&= n\left(tr(\widehat{\boldsymbol{\Sigma}}\boldsymbol{\Sigma}(\boldsymbol{\theta})^{-1}) + \ln|\boldsymbol{\Sigma}(\boldsymbol{\theta})| - \ln|\widehat{\boldsymbol{\Sigma}}| - k\right)
\end{aligned}
$$

- Actually, minimize the "Objective Function"

$$
tr(\widehat{\boldsymbol{\Sigma}}\boldsymbol{\Sigma}(\boldsymbol{\theta})^{-1}) + \ln|\boldsymbol{\Sigma}(\boldsymbol{\theta})| - \ln|\widehat{\boldsymbol{\Sigma}}| - k
$$

And multiply by n to get the LR test statistic

Saturated Models

- If there are the same number of moment structure equations and unknown parameters and the parameter is identifiable, there is a one-to-one function between $\widehat{\boldsymbol{\Sigma}} \text{ and } \widehat{\boldsymbol{\theta}}$
- Sometimes called "just identifiable."
- In this case, the model imposes NO equality constraints on Sigma.
- G=0, df=0 and the standard test for goodness of fit does not apply. The model is un-testable.
- Or is it? There could still be inequality constraints.