Double measurement with correlated measurement error

$$
\begin{array}{rcl}
Y_i & = & \beta_0 + \beta_1 X_i + \epsilon_i \\
W_{i,1} & = & \nu_1 + X_i + e_{i,1} \\
W_{i,2} & = & \nu_2 + X_i + e_{i,2},
\end{array}
$$

$$Cov(e_{i,1}, e_{i,2}) = \omega_{1,2}$$

$$
\boldsymbol{\Sigma} = [\sigma_{i,j}] = \left[\begin{array}{ccc}
\phi + \omega_1 & \phi + \omega_{1,2} & \beta_1 \phi \\
 & \phi + \omega_2 & \beta_1 \phi \\
 & & \beta_1^2 \phi + \psi
\end{array} \right]
$$

$$\mathbf{\Sigma} = [\sigma_{i,j}] = \left[\begin{array}{ccc} \phi + \omega_1 & \phi + \omega_{1,2} & \beta_1 \phi \\ & \phi + \omega_2 & \beta_1 \phi \\ & & \beta_1^2 \phi + \psi \end{array} \right]$$

Let $\mathbf{\Sigma}$ be any symmetric positive definite matix with $\sigma_{1,3} = \sigma_{2,3}$, and let $\omega_{1,2}$ range over $(-\infty, \sigma_{1,2})$. Now let all the other parameters depend on $\omega_{1,2}$ as follows:

$$\phi = \sigma_{1,2} - \omega_{1,2}$$

$$\omega_1 = \sigma_{1,1} - \sigma_{1,2} + \omega_{1,2}$$

$$\omega_2 = \sigma_{2,2} - \sigma_{1,2} + \omega_{1,2}$$

$$\beta_1 = \frac{\sigma_{1,3}}{\sigma_{1,2} - \omega_{1,2}}$$

$$\psi = \sigma_{3,3} - \frac{\sigma_{1,3}^2}{\sigma_{1,2} - \omega_{1,2}}$$

And the covariance matrix remains constant.

Infinitely many sets of parameter values yield the same covariance matrix

- And hence the same distribution of the observable data.
- The sets yielding the same distribution form a curvy 5-d surface in the 6-d parameter space.
- For the set of covariances with $\sigma_{13}=\sigma_{23}$ that give the highest likelihood, we have an MLE (over Θ) that is not unique.
- Maximum likelihood estimation will fail

But still there is useful information in **Σ**-hat

- There is a testable equality constraint
- $\omega_{12} < \sigma_{12}$
- Can tell if β_1 is positive, negative or zero

$$\boldsymbol{\Sigma} = \left[\begin{array}{ccc} \phi + \omega_1 & \phi + \omega_{1,2} & \beta_1 \phi \\ & \phi + \omega_2 & \beta_1 \phi \\ & & \beta_1^2 \phi + \psi \end{array} \right]$$