The Multivariate Normal Distribution

The $p \times 1$ random vector $\mathbf{X}$ is said to have a *multivariate normal distribution*, and we write $\mathbf{X} \sim N(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, if $\mathbf{X}$ has (joint) density

$$f(\mathbf{x}) = \frac{1}{|\boldsymbol{\Sigma}|^{\frac{1}{2}}(2\pi)^{\frac{p}{2}}} \exp\left[-\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\mu})\right],$$

where $\boldsymbol{\mu}$ is $p \times 1$ and $\boldsymbol{\Sigma}$ is $p \times p$ symmetric and positive definite. Positive definite means that for any non-zero $p \times 1$ vector $\mathbf{a}$, we have $\mathbf{a}'\boldsymbol{\Sigma}\mathbf{a} > 0$.

- Since the one-dimensional random variable $Y = \sum_{i=1}^{p} a_i X_i$ may be written as $Y = \mathbf{a}'\mathbf{X}$ and $Var(Y) = V(\mathbf{a}'\mathbf{X}) = \mathbf{a}'\boldsymbol{\Sigma}\mathbf{a}$, it is natural to require that $\boldsymbol{\Sigma}$ be positive definite. All it means is that every non-zero linear combination of $\mathbf{X}$ values has a positive variance.
- $\boldsymbol{\Sigma}$ positive definite is equivalent to $\boldsymbol{\Sigma}^{-1}$ positive definite.

The multivariate normal reduces to the univariate normal when $p = 1$. Other properties of the multivariate mormal include the following.

1. $E(\mathbf{X}) = \boldsymbol{\mu}$
2. $V(\mathbf{X}) = \boldsymbol{\Sigma}$
3. If $\mathbf{c}$ is a vector of constants, $\mathbf{X} + \mathbf{c} \sim N(\mathbf{c} + \boldsymbol{\mu}, \boldsymbol{\Sigma})$
4. If $\mathbf{A}$ is a matrix of constants, $\mathbf{AX} \sim N(\mathbf{A}\boldsymbol{\mu}, \mathbf{A}\boldsymbol{\Sigma}\mathbf{A}')$
5. All the marginals (dimension less than p) of $\mathbf{X}$ are (multivariate) normal, but it is possible in theory to have a collection of univariate normals whose joint distribution is not multivariate normal.
6. For the multivariate normal, zero covariance implies independence. The multivariate normal is the only continuous distribution with this property.
7. The random variable $(\mathbf{X}-\boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\mathbf{X}-\boldsymbol{\mu})$ has a chi-square distribution with p degrees of freedom.
8. After a bit of work, the multivariate normal likelihood may be written as

$$L(\boldsymbol{\mu}, \boldsymbol{\Sigma}) = |\boldsymbol{\Sigma}|^{-n/2}(2\pi)^{-np/2} \exp -\frac{n}{2}\left[tr(\widehat{\boldsymbol{\Sigma}}\boldsymbol{\Sigma}^{-1}) + (\overline{\mathbf{x}}-\boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\overline{\mathbf{x}}-\boldsymbol{\mu})\right],$$

where $\widehat{\boldsymbol{\Sigma}} = \frac{1}{n}\sum_{i=1}^{n}(\mathbf{x}_i - \overline{\mathbf{x}})(\mathbf{x}_i - \overline{\mathbf{x}})'$ is the sample variance-covariance matrix (it would be unbiased if divided by $n-1$).