STA 413 Formulas

Distribution	$f(x)$	$M(t)$	$E(X)$	$Var(X)$
Bernoulli	$\theta^x(1-\theta)^{1-x}I(x=0,1)$	$\theta e^t+1-\theta$	θ	$\theta(1-\theta)$
Binomial	$\binom{n}{x}\theta^x(1-\theta)^{n-x}I(x=0,\ldots,n)$	$(\theta e^t+1-\theta)^n$	$n\theta$	$n\theta(1-\theta)$
Poisson	$\frac{e^{-\lambda}\lambda^x}{x!}I(x=0,1,\ldots)$	$e^{\lambda(e^t-1)}$	λ	λ
Geometric	$\theta(1-\theta)^{x-1}I(x=1,2,\ldots)$	$\theta(e^{-t}+\theta-1)^{-1}$	$\frac{1}{\theta}$	$\frac{1-\theta}{\theta^2}$
Exponential	$\frac{1}{\theta}e^{-x/\theta}I(x>0)$	$(1-\theta t)^{-1}$	θ	θ^2
Gamma	$\frac{1}{\beta^\alpha\Gamma(\alpha)}e^{-x/\beta}x^{\alpha-1}I(x>0)$	$(1-\beta t)^{-\alpha}$	$\alpha\beta$	$\alpha\beta^2$
Chi-square	$\frac{1}{2^{\nu/2}\Gamma(\nu/2)}e^{-x/2}x^{\nu/2-1}I(x>0)$	$(1-2t)^{-\nu/2}$	ν	2ν
Normal	$\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{(x-\mu)^2}{2\sigma^2}}$	$e^{\mu t+\frac{1}{2}\sigma^2t^2}$	μ	σ^2
Uniform	$\frac{1}{\beta-\alpha}I(\alpha\le x\le\beta)$	$\frac{e^{\beta t}-e^{\alpha t}}{t(\beta-\alpha)}$	$\frac{\alpha+\beta}{2}$	$\frac{(\beta-\alpha)^2}{12}$
Beta	$\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}x^{\alpha-1}(1-x)^{\beta-1}I(0<x<1)$		$\frac{\alpha}{\alpha+\beta}$	$\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$

$$e^x=\sum_{k=0}^{\infty}\frac{x^k}{k!}$$

$$\sum_{k=r}^{\infty}a^k=\frac{a^r}{1-a}\text{ for }0<a<1$$

$E[g(X)]=\sum_x g(x)p(x)$

$E[g(X)]=\int_{-\infty}^{\infty}g(x)f(x)\,dx$

$F_{Y_1}(y)=1-[1-F(y)]^n$

$f_{Y_1}(y)=n[1-F(y)]^{n-1}f(y)$

$F_{Y_n}(y)=[F(y)]^n$

$f_{Y_n}(y)=n[F(y)]^{n-1}f(y)$

$$S_n^2=\frac{\sum_{i=1}^n(X_i-\overline{X})^2}{n-1}=\frac{1}{n-1}\left(\sum_{i=1}^n X_i^2-n\overline{X}^2\right)$$

$E(g(X))\ge aPr\{g(X)\ge a\}$ for $g(x)\ge 0$

If $A_1\subseteq A_2\subseteq\ldots$ then $\lim_{n\to\infty}P(A_n)=P\left(\bigcup_{n=1}^{\infty}A_n\right)$

If $A_1\supseteq A_2\supseteq\ldots$ then $\lim_{n\to\infty}P(A_n)=P\left(\bigcap_{n=1}^{\infty}A_n\right)$

$X_n\xrightarrow{a.s.}X$ means $P\{c:\lim_{n\to\infty}X_n(c)=X(c)\}=1$.

$X_n\xrightarrow{p}X$ means for all $\epsilon>0$, $\lim_{n\to\infty}P\{|X_n-X|<\epsilon\}=1$.

$X_n\xrightarrow{d}X$ means for every continuity point x of F_X, $\lim_{n\to\infty}F_{X_n}(x)=F_X(x)$.

$X_n\xrightarrow{a.s.}X\Rightarrow X_n\xrightarrow{p}X\Rightarrow X_n\xrightarrow{d}X$

If a is a constant, $X_n\xrightarrow{d}a\Rightarrow X_n\xrightarrow{p}a$.

$\overline{X}_n\xrightarrow{a.s.}\mu$

Taylor's Theorem (Just for two terms plus remainder): Let $g(x)$ be a function with $g''(x)$ continuous at $x=x_0$. Then

$$g(x)=g(x_0)+g'(x_0)(x-x_0)+\frac{g''(x^*)(x-x_0)^2}{2!},$$

where x^* is between x and x_0.

Convergence Rules

1. Convergence in Probability
   (a) If $X_n \xrightarrow{p} X$ and $Y_n \xrightarrow{p} Y$ then $X_n + Y_n \xrightarrow{p} X + Y$.
   (b) If $X_n \xrightarrow{p} X$ then $aX_n \xrightarrow{p} aX$.
   (c) If $X_n \xrightarrow{p} X$ and the function $g(x)$ is continuous except possibly on a set A with $P\{X \in A\} = 0$, then $g(X_n) \xrightarrow{p} g(X)$.
   (d) If $X_n \xrightarrow{p} X$ and $Y_n \xrightarrow{p} Y$ then $X_n Y_n \xrightarrow{p} XY$.
   (e) *Variance Rule*: If $\lim_{n\to\infty} E(T_n) = \theta$ and $\lim_{n\to\infty} Var(T_n) = 0$, then $T_n \xrightarrow{p} \theta$.
   (f) *Weak Law of Large Numbers*: $\overline{X}_n \xrightarrow{p} \mu$
2. Convergence in Distribution
   (a) If $X_n \xrightarrow{d} X$ and the function $g(x)$ is continuous except possibly on a set A with $P\{X \in A\} = 0$, then $g(X_n) \xrightarrow{d} g(X)$.
   (b) If $X_n \xrightarrow{d} X$, $A_n \xrightarrow{p} a$ and $B_n \xrightarrow{p} b$, then $A_n + B_n X_n \xrightarrow{d} a + bX$.
   (c) *Central Limit Theorem*: If $X_1, \ldots, X_n$ are a random sample from a disribution with expected value μ and variance σ^2, then $\sqrt{n}(\overline{X}_n - \mu) \xrightarrow{d} X \sim N(0, \sigma^2)$.
   (d) *Delta method*: If $T_n \xrightarrow{p} c$, $\sqrt{n}(T_n - c) \xrightarrow{d} T$, and $g(x)$ is a function with $g'(c) \neq 0$ and $g''(x)$ continuous at $x = c$, then $\sqrt{n}(g(T_n) - g(c)) \xrightarrow{d} g'(c)T$.
   (e) *Delta method combined with CLT*: $\sqrt{n}(g(\overline{X}_n) - g(\mu)) \xrightarrow{d} Y \sim N(0, g'(\mu)^2\sigma^2)$.

Regularity Conditions: Let $X_1, \ldots, X_n \overset{i.i.d.}{\sim} f(x|\theta), \theta \in \Omega$

R0: If $\theta_1 \neq \theta_2$, cannot have $f(x|\theta_1) = f(x|\theta_2)$ for all x.

R1: The support of $f(x|\theta)$ does not depend on θ.

R2: Ω is an open set, so that each $\theta \in \Omega$ is surrounded by a neighborhood of points in Ω.

R3: $\frac{\partial^2}{\partial\theta^2} f(x|\theta)$ exists.

R4: $\frac{\partial^2}{\partial\theta^2} \int f(x|\theta)\,dx = \int \frac{\partial^2}{\partial\theta^2} f(x|\theta)\,dx$.

R5: $\frac{\partial^3}{\partial\theta^3} f(x|\theta)$ exists. And for all $\theta \in \Omega$, there exists a function $M(x)$ and a constant c such that $\left|\frac{\partial^3}{\partial\theta^3} \log f(x|\theta)\right| \leq M(x)$ with $E_{\theta_0}(M(X)) < \infty$ for all $\theta_0 - c < \theta < \theta_0 + c$ and all x in the support of $f(x|\theta)$. True parameter is θ_0.

Likelihood Formulas

$I(\theta) = -E\left(\frac{\partial^2}{\partial\theta^2} \ln f(X;\theta)\right) = E\left[\frac{\partial}{\partial\theta} \ln f(X;\theta)\right]^2$

$S = \frac{\partial}{\partial\theta} \ln f(X;\theta)$. $E(S) = 0$, $Var(S) = I(\theta)$.

If $E(T) = \theta$, $Var(T) \geq \frac{1}{nI(\theta)}$

$Y_n = \sqrt{n}(\widehat{\theta}_n - \theta_0) \xrightarrow{d} Y \sim N(0, \frac{1}{I(\theta)})$

$C = \{\mathbf{x} : \Lambda_n < k\}$, where $\Lambda_n = \frac{L(\theta_0)}{L(\widehat{\theta}_n)}$

$G_n = -2\ln\Lambda_n = 2(\ell(\widehat{\theta}) - \ell(\widehat{\theta}_0)) \xrightarrow{d} G \sim \chi^2(r)$

$L(\theta, \mathbf{x}) = g(t, \theta)h(\mathbf{x})$

$h(\theta|\mathbf{x}) \propto L(\theta, \mathbf{x})h(\theta)$