STA 413F2011 Assignment 3

Do this assignment in preparation for Quiz Three in tutorial on Friday Sept 30th. The problems are practice for the quiz, and are not to be handed in. See Formula Sheet 3; a copy will be supplied with the quiz.

1. Let X be a discrete random variable with $P(X = \theta) = 1$, where θ is a real number.
   (a) Find the moment-generating function of X; show your work.
   (b) Use the moment-generating function to find $E(X)$; show your work.
2. Let $T_1, T_2, \ldots$ be a sequence of real-valued random variables. Show that if $\lim_{n\rightarrow\infty} E(T_n) = \theta$ and $\lim_{n\rightarrow\infty} Var(T_n) = 0$, then $T_n \stackrel{P}{\rightarrow} \theta$. Hint: Use Markov's inequality.
3. Let $P(T_n = 0) = \frac{n-1}{n}$ and $P(T_n = n^2) = \frac{1}{n}$.
   (a) Show $T_n \stackrel{P}{\rightarrow} 0$ using the definition of convergence in probability.
   (b) Does $E(T_n) \rightarrow 0$? Answer Yes or No and prove your answer.
   (c) Does $Var(T_n) \rightarrow 0$? Answer Yes or No and prove your answer.

   This shows you that the Variance Rule is a sufficient but not a necessary condition for convergence in probability.
4. Do Problem 4.2.2. For Part (a), use the variance rule. For (b) and (c), you are allowed to use theorems from the text.
5. Do Problem 4.2.3. Something you've already proved is more helpful than Chebyshev's inequality.
6. Do Problem 4.2.4.
7. Do Problem 4.2.5.
8. Let $X_1, \ldots, X_n$ be a random sample from a distribution with expected value μ and variance σ^2. The Law of Large Numbers says $\overline{X}_n \stackrel{P}{\rightarrow} \mu$. Show that $\overline{X}_n + a \stackrel{P}{\rightarrow} \mu + a$, where a is a constant. Use the definition of convergence in probability.
9. As before, let $X_1, \ldots, X_n$ be a random sample from a distribution with expected value μ and variance σ^2. Prove that $3\overline{X}_n \stackrel{P}{\rightarrow} 3\mu$. Use the definition of convergence in probability.
10. Let $X_1, \ldots, X_n$ be a random sample from a distribution with expected value μ and variance σ^2_x. Independently of $X_1, \ldots, X_n$, let $Y_1, \ldots, Y_n$ be a random sample from a distribution with the same expected value μ and variance σ^2_y. Let $T_n = \alpha\overline{X}_n + (1-\alpha)\overline{Y}_n$, where α is a constant between zero and one.

(a) How do you know $\overline{X}_n$ and $\overline{Y}_n$ are independent?

(b) Is T_n an unbiased estimator of μ? Answer Yes or No and show your work.

(c) Is T_n a consistent estimator of μ? Answer Yes or No and show your work.

(d) What value of α would make the estimator T_n most accurate in terms of having the smallest possible variance? Show your work.

11. Let $X_1, \ldots, X_n$ be independent random variables with a continuous uniform distribution on $[0, \theta]$, and let $Y_n = \max(X_1, \ldots, X_n)$.

    (a) Using the definition of convergence in probability, show that Y_n is a consistent estimator of θ.

    (b) Show that $2\overline{X}_n$ is also consistent for θ.

12. Let X be a random variable with expected value μ and variance σ^2. Show that $\frac{X}{n} \stackrel{P}{\rightarrow} 0$.

13. Let $X_1, \ldots, X_n$ be a random sample from an exponential distribution with parameter θ, and let $T_n = nY_1$, where $Y_1 = \min(X_1, \ldots, X_n)$.

    (a) Find the probability density function of T_n. Don't forget the support.

    (b) Is T_n unbiased? Answer Yes or No. You can use the formula sheet.

    (c) Is T_n consistent? Answer Yes or No and justify your answer by giving an explicit formula for $P(|T_n - \theta| < \epsilon)$.

14. A model for simple regression through the origin is

$$Y_i = \beta x_i + \epsilon_i$$

for $i = 1, \ldots, n$, where $\epsilon_1, \ldots, \epsilon_n$ are a random sample from a distribution with expected value zero and variance σ^2, and β and σ^2 are unknown constants.

    (a) What is $E(Y_i)$?

    (b) What is $Var(Y_i)$?

    (c) Find the Least Squares estimate of β by minimizing $Q = \sum_{i=1}^{n}(Y_i - \beta x_i)^2$ over all values of β. Let $\widehat{\beta}_n$ denote the point at which Q is minimal.

    (d) Is $\widehat{\beta}_n$ unbiased? Answer Yes or No and show your work.

    (e) Give a sufficient condition for $\widehat{\beta}_n$ to be consistent. Show your work. Remember, in this model the x_i are fixed constants, not random variables.

    (f) Let $\widehat{\beta}_{2,n} = \frac{\overline{Y}_n}{\overline{x}_n}$. Is $\widehat{\beta}_{2,n}$ unbiased? Consistent? Answer Yes or No to each question and show your work.

    (g) Prove that $\widehat{\beta}_n$ is a more accurate estimator than $\widehat{\beta}_{2,n}$ in the sense that it has smaller variance. Hint: The sample variance of the independent variable values cannot be negative.