STA 413F2011 Assignment 10

1. Let $X_1, \ldots, X_n$ be a random sample from a distribution with density $f(x;\theta_0)$.

   (a) The random variable
   $$\frac{1}{n}\sum_{i=1}^n \frac{\partial}{\partial\theta} \ln f(X_i;\theta)$$
   converges in probability to something. What is the target? How do you know?

   (b) The random variable
   $$-\frac{1}{n}\sum_{i=1}^n \frac{\partial^2}{\partial\theta^2} \ln f(X_i;\theta)$$
   converges in probability to something. What is the target? How do you know?

   (c) Write down a Taylor expansion (two terms plus remainder) of the function $\ell'(\widehat{\theta}_n)$; expand about the true parameter θ_0.

   (d) Assuming the remainder term goes to zero in probability (it does), find the limiting distribution of $\sqrt{n}(\widehat{\theta}_n - \theta_0)$. Cite facts from the formula sheet and homework assignments as you use them.

   (e) Prove $\sqrt{nI(\widehat{\theta}_n)}\,(\widehat{\theta}_n - \theta_0) \stackrel{d}{\rightarrow} Z \sim N(0,1)$, citing facts from the formula sheet as you use them. It is okay to assume that the function $I(t)$ is continuous.

   (f) Derive a $(1-\alpha)100\%$ confidence interval for θ. Show all your work.

2. Let $X_1, \ldots, X_n$ be a random sample from a distribution with density
   $$f(x;\theta) = (\theta+1)x^\theta I(0 < x < 1),$$
   where $\theta > 0$.

   (a) Verify
   $$\mu = \frac{\theta+1}{\theta+2} \quad \text{and} \quad \sigma^2 = \frac{\theta+1}{(\theta+3)(\theta+2)^2}.$$

   (b) Let
   $$T_n = \frac{2\overline{X}_n - 1}{1 - \overline{X}_n}.$$
   Is T_n a consistent estimator of θ? Answer Yes or No and prove your answer, citing facts from the formula sheet as you use them.

   (c) Use the delta method to show $\sqrt{n}(T_n - \theta_0) \stackrel{d}{\rightarrow} X$. What is the distribution of X? Give its mean and variance.

   (d) Obtain a formula for the MLE of θ. Show your work.

   (e) Calculate the Fisher Information $I(\theta)$. Show your work.

   (f) $\sqrt{n}(\widehat{\theta}_n - \theta) \stackrel{d}{\rightarrow} Y$. What is the distribution of Y? Give its mean and variance.

   (g) Compare $Var(X)$ from Question 2c to $Var(Y)$ from Question 2f. Which is smaller? Prove your answer. Of course smaller is better.

(h) A random sample of size $n = 50$ yields $\sum_{i=1}^{n} X_i = 39.557$ and $\sum_{i=1}^{n} \ln(X_i) = -12.947$.

i. What is T_n? Your answer is a single number.

ii. What is $\widehat{\theta}_n$? Your answer is a single number.

iii. Give an approximate 95% confidence interval for θ based on T_n. Your answer is a pair of numbers.

iv. Give an approximate 95% confidence interval for θ based on $\widehat{\theta}_n$. Your answer is a pair of numbers.

3. Let $X_1, \ldots, X_n$ be a random sample from a $N(\theta, \sigma^2)$ distribution, with σ^2 known.

(a) Derive an *exact* likelihood ratio test of $H_0 : \theta = \theta_0$. Show your work.

(b) Suppose $\sigma^2 = 4$, $\alpha = 0.05$, the null hypothesis is $H_0 : \theta = 0$, and we observe $\overline{X}_n = -1.2$ with $n = 9$. Do you reject H_0? Answer Yes or No and show all your work.

(c) Carry out a large-sample likelihood ratio test for this problem. Comment.

4. Do problem 6.3.5. The problem is asking for an exact likelihood ratio test. Calcuate and simplify the large sample likelihood ratio test statistic as well.

5. Let $X_1, \ldots, X_n$ be a random sample from a Bernoulli distribution with parameter θ. The goal is to test $H_0 : \theta = \theta_0$ versus $H_0 : \theta \neq \theta_0$. Suppose $\alpha = 0.05, n = 100, \theta_0 = \frac{1}{2}$ and $\overline{X}_n = 0.6$.

(a) Carry out a large-sample likelihood ratio test for this problem. Your final answer is a test statistic (a *number*) and a statement of whether you reject the null hypothesis, Yes or No.

(b) Carry out a common-sense Z-test based on the central limit theorem. Compare results.

6. Do problem 6.3.10.

Bring a calculator to the quiz!