STA 413F2008 Assignment 8

The following questions are practice for Test 3 and the Final Examination. They are not to be handed in.

1. Let $X_1, \ldots, X_n$ be a random sample from a Poisson distribution with parameter λ. Consider the following critical region for testing $H_0 : \lambda \geq \lambda_0$ versus $H_1 : \lambda < \lambda_0$:

$$C = \left\{ (x_1, \ldots, x_n) : \frac{\sqrt{n}(\overline{x}_n - \lambda_0)}{\sqrt{\lambda_0}} < -z_\alpha \right\}$$

   (a) Derive a formula for the power function $P_\lambda(\mathbf{X} \in C)$. "Derive" means show all your work.

   (b) For $\alpha = 0.05$, $n = 50$ and $\lambda_0 = 4$, what is the approximate power of the test at $\lambda = 3$? I get 0.539.

   (c) What is ω_0 for this problem?

   (d) What is ω_1 for this problem?

   (e) What is the approximate distribution of the test statistic when $\lambda = \lambda_0$? Cite the formula sheet to support your answer.

   (f) If $\alpha = 0.05$, and $\lambda_0 = 4$, what is the minimum sample size required so that the approximate power of the test will be at least 0.80 at $\lambda = 3$?

2. Let $X_1, \ldots, X_n$ be a random sample from an exponential distribution with parameter θ. Consider the following critical region for testing $H_0 : \theta \leq \theta_0$ versus $H_1 : \theta > \theta_0$:

$$C_1 = \left\{ (x_1, \ldots, x_n) : \sqrt{n} \left(\frac{\overline{x}_n}{\theta_0} - 1 \right) > z_\alpha \right\}$$

   (a) Derive a formula for the power function $P_\theta(\mathbf{X} \in C_1)$. "Derive" means show all your work.

   (b) For $\alpha = 0.05$, $n = 50$ and $\theta_0 = 4$, what is the approximate power of the test at $\theta = 5$?

   (c) What is ω_0 for this problem?

   (d) What is ω_1 for this problem?

   (e) For $\theta \in \omega_0$, does the power function attain its maximum at $\theta = \theta_0$? Answer Yes or No and prove your answer. A picture may help.

   (f) What is the approximate distribution of the test statistic when $\theta = \theta_0$? Show some calculations and cite something from the formula sheet to support your answer.

   (g) Does this mean the test is of (approximate) size α? Answer Yes or No.

   (h) If $\alpha = 0.05$, and $\theta_0 = 4$, what is the minimum sample size required so that the approximate power of the test will be at least 0.80 at $\theta = 5$? Your answer is a single number. My answer is 117.

3. Consider the exponential example of Question 2, except now we want to test $H_0 : \theta = \theta_0$ versus $H_1 : \theta \neq \theta_0$. The critical region will be

$$C_2 = \left\{ (x_1, \ldots, x_n) : \sqrt{n} \left| \frac{\overline{x}_n}{\theta_0} - 1 \right| > z_{\alpha/2} \right\}.$$

   (a) Derive a formula for the power function $P_\theta(\mathbf{X} \in C_2)$. "Derive" means show all your work.

   (b) If $\alpha = 0.05$, and $\theta_0 = 4$, what is the minimum sample size required so that the approximate power of the test will be at least 0.80 at $\theta = 5$? It's not easy to solve for n here, so the best strategy is to try a few values. I suggest starting with your answer to 2h.

   (c) So, in terms of sample size, what is the price you must pay here for using a two-sided test instead of one-sided? Your answer is a single number. My answer is 29.

4. Let $X_1, \ldots, X_n$ be a random sample from a Gamma distribution with parameters α (unknown) and $\beta = 1$ (known). Using just the Modified Central Limit Theorem (no variance-stabilizing transformations, please!), give an approximate critical region of size 0.05 for the following null and alternative hypotheses. Please use specific numbers for your critical values, not symbols. You don't have to prove anything or justify your answers. But you do have to be aware of the justification in order to get the right answer.

   (a) $H_0 : \alpha = \alpha_0$ versus $H_1 : \alpha \neq \alpha_0$.

   (b) $H_0 : \alpha = \alpha_0$ versus $H_1 : \alpha > \alpha_0$.

   (c) $H_0 : \alpha \leq \alpha_0$ versus $H_1 : \alpha > \alpha_0$.

   (d) $H_0 : \alpha = \alpha_0$ versus $H_1 : \alpha < \alpha_0$.

   (e) $H_0 : \alpha \geq \alpha_0$ versus $H_1 : \alpha < \alpha_0$.

5. For Question 4, suppose $n = 150$, $\alpha_0 = 7.5$ and you observe a sample mean of $\overline{X}_n = 8.2$. Test $H_0 : \alpha \leq \alpha_0$ versus $H_1 : \alpha > \alpha_0$.

   (a) What is the value of the test statistic? The answer is a single number.

   (b) What is the p-value? The answer is a single number.

   (c) Do you reject H_0? Answer Yes or No.

   (d) Is $p < \alpha$? Answer Yes or No.

6. Let $X_1, \ldots, X_n$ be a random sample from a Binomial distribution with parameters $m = 10$ (known) and θ (unknown). Using just the Modified Central Limit Theorem (no variance-stabilizing transformations, please!), give an approximate critical region of size 0.05 for the following null and alternative hypotheses. Please use specific numbers for your critical values, not symbols. You don't have to prove anything or justify your answers. But you do have to be aware of the justification in order to get the right answer.

   (a) $H_0 : \theta = \theta_0$ versus $H_1 : \theta \neq \theta_0$.

   (b) $H_0 : \theta = \theta_0$ versus $H_1 : \theta > \theta_0$.

   (c) $H_0 : \theta \leq \theta_0$ versus $H_1 : \theta > \theta_0$.

   (d) $H_0 : \theta = \theta_0$ versus $H_1 : \theta < \theta_0$.

   (e) $H_0 : \theta \geq \theta_0$ versus $H_1 : \theta < \theta_0$.

7. For Question 6, suppose $n = 150$, $\theta_0 = 0.75$ and you observe a sample mean of $\overline{X}_n = 8.2$. Test $H_0 : \theta \geq \theta_0$ versus $H_1 : \theta < \theta_0$.

   (a) What is the value of the test statistic? The answer is a single number.

   (b) What is the p-value? The answer is a single number.

   (c) Do you reject H_0? Answer Yes or No.

   (d) Is $p < \alpha$? Answer Yes or No.

8. Let $X_1, \ldots, X_n$ be a random sample from a Geometric distribution with parameter θ. Using just the Modified Central Limit Theorem (no variance-stabilizing transformations, please!), give an approximate critical region of size 0.05 for the following null and alternative hypotheses. Please use specific numbers for your critical values, not symbols. You don't have to prove anything or justify your answers. But you do have to be aware of the justification in order to get the right answer.

   (a) $H_0 : \theta = \theta_0$ versus $H_1 : \theta \neq \theta_0$.

   (b) $H_0 : \theta = \theta_0$ versus $H_1 : \theta > \theta_0$.

   (c) $H_0 : \theta \leq \theta_0$ versus $H_1 : \theta > \theta_0$.

   (d) $H_0 : \theta = \theta_0$ versus $H_1 : \theta < \theta_0$.

   (e) $H_0 : \theta \geq \theta_0$ versus $H_1 : \theta < \theta_0$.

9. For Question 8, suppose $n = 150$, $\theta_0 = 0.10$ and you observe a sample mean of $\overline{X}_n = 8.2$. Test $H_0 : \theta = \theta_0$ versus $H_1 : \theta \neq \theta_0$.

   (a) What is the value of the test statistic? The answer is a single number.

   (b) What is the p-value? The answer is a single number.

   (c) Do you reject H_0? Answer Yes or No.

   (d) Is $p < \alpha$? Answer Yes or No.

10. Let $X_1, \ldots, X_n$ be a random sample from a Normal distribution with parameters μ and σ^2, both unknown. Using just the Modified Central Limit Theorem (no variance-stabilizing transformations, please!), give an approximate critical region of size $\alpha = 0.05$ for the following null and alternative hypotheses. Please use specific numbers for your critical values, not symbols. You don't have to prove anything or justify your answers. But you do have to be aware of the justification in order to get the right answer.

   (a) $H_0 : \mu = \mu_0$ versus $H_1 : \mu \neq \mu_0$.
   (b) $H_0 : \mu = \mu_0$ versus $H_1 : \mu > \mu_0$.
   (c) $H_0 : \mu \leq \mu_0$ versus $H_1 : \mu > \mu_0$.
   (d) $H_0 : \mu = \mu_0$ versus $H_1 : \mu < \mu_0$.
   (e) $H_0 : \mu \geq \mu_0$ versus $H_1 : \mu < \mu_0$.

11. For Question 10, what critical values would you use to make the tests exact? Use symbols rather than specific numbers, please.

   (a) $H_0 : \mu = \mu_0$ versus $H_1 : \mu \neq \mu_0$.
   (b) $H_0 : \mu = \mu_0$ versus $H_1 : \mu > \mu_0$.
   (c) $H_0 : \mu \leq \mu_0$ versus $H_1 : \mu > \mu_0$.
   (d) $H_0 : \mu = \mu_0$ versus $H_1 : \mu < \mu_0$.
   (e) $H_0 : \mu \geq \mu_0$ versus $H_1 : \mu < \mu_0$.

12. Let $Y_i = x_i + \epsilon_i$, for $i = 1, \ldots, n$, where

   - $x_1, \ldots, x_n$ are fixed, known constants
   - $\epsilon_1, \ldots, \epsilon_n$ are independent and identically distributed Normal$(0,\sigma^2)$ random variables; the parameter σ^2 is unknown.
   - The data consist of n pairs (x_i, Y_i). Of course the error terms ϵ_i are not observable.

   (a) Find the distribution of $\frac{1}{\sigma^2}\sum_{i=1}^n (Y_i - x_i)^2$. Show your work.
   (b) We wish to test $H_0 : \sigma^2 \leq \sigma_0^2$ against $H_1 : \sigma^2 > \sigma_0^2$.
      i. What is Ω?
      ii. What is ω_0? Is it simple or composite?
      iii. What is ω_1? Is it simple or composite?
   (c) Find the constant k so that the following test will be size α for testing the *simple* null hypothesis $H_0 : \sigma^2 = \sigma_0^2$.

   $$C = \{\mathbf{y} : \frac{\sum_{i=1}^n (y_i - x_i)^2}{\sigma_0^2} > k\}$$

   (d) Find the power function $P_{\sigma^2}\{\mathbf{X} \in C\} = \pi(\sigma^2)$.

(e) Prove that the test C is also size α for testing $H_0 : \sigma^2 \leq \sigma_0^2$.

13. In this unfamiliar but reasonable regression model, the effect of x is linear as usual, but each member of the population has his or her own individual slope. That makes the slope a random variable, because if you took another sample, you'd get another collection of slopes. Accordingly,

    Let $Y_i = x_i B_i$, for $i = 1, \ldots, n$, where

    - $x_1, \ldots, x_n$ are fixed, known constants
    - $B_1, \ldots, B_n$ are independent and identically distributed Normal(β, σ^2) random variables.
    - The parameters β and σ^2 are unknown.
    - The data consist of n pairs (x_i, Y_i).
    - Yes, there is no error term. Don't worry.

    We wish to test $H_0 : \beta = \beta_0$ against $H_1 : \beta \neq \beta_0$.

    (a) What is Ω?

    (b) What is ω_0? Is it simple or composite?

    (c) What is ω_1? Is it simple or composite?

    (d) Find the distribution of $\frac{1}{n}\sum_{i=1}^n \frac{Y_i}{x_i}$.

    (e) Find the constant k so that the following test will be size α:

$$C = \left\{ \mathbf{y} : \left| \frac{\sqrt{n(n-1)}[(\frac{1}{n}\sum_{i=1}^n \frac{y_i}{x_i}) - \beta_0]}{\sqrt{\sum_{i=1}^n [y_i/x_i - (\frac{1}{n}\sum_{j=1}^n \frac{y_j}{x_j})]^2}} \right| > k \right\}$$

    i. Approximately

    ii. Exactly

14. Let $X_1, \ldots, X_{n_1}$ be a random sample from a uniform distribution on $(0, \theta]$. We wish to test $H_0 : \theta = \theta_0$ against $H_1 : \theta \neq \theta_0$..

   (a) Write down the cumulative distribution function of the sample maximum Y_n; you will need it.

   (b) Show that the following test is of size α:

$$C = \left\{ \mathbf{x} : Y_n > \theta_0 \text{ or } Y_n < \theta_0 \alpha^{1/n} \right\}$$

   (c) It makes sense that the critical value $\theta_0 \alpha^{1/n}$ is less than θ_0, but please prove it. This will help later.

   (d) Now we will find the power function $P_\theta\{\mathbf{X} \in C\} = \pi(\theta)$

      i. Show $\pi(\theta) = 1$ for $\theta < \theta_0 \alpha^{1/n}$.

      ii. Show $\pi(\theta) = (\frac{\theta_0}{\theta})^n \alpha$ for $\theta_0 \alpha^{1/n} \leq \theta < \theta_0$.

      iii. That looks suspicious because $\frac{\theta_0}{\theta} > 1$. To see that it's okay, show that $(\frac{\theta_0}{\theta})^n \alpha \leq 1$ for $\theta_0 \alpha^{1/n} \leq \theta < \theta_0$.

      iv. Show $\pi(\theta) = 1 - (\frac{\theta_0}{\theta})^n (1 - \alpha)$ for $\theta \geq \theta_0$.

   (e) Suppose that $\theta_0 = 10$, $\alpha = 0.05$, and the true value of θ is 9. What value of n is required so that the probability of Type II Error is zero? My answer is $n = 29$.

   (f) Suppose the true value of θ is *less than* 9 and $n = 29$. Is the probability of Type II Error still zero?

   (g) Suppose $\theta > \theta_0$. What sample size is required so that the probability of Type II error is less than β, where $0 < \beta < 1$? My answer is $n > \frac{\ln \theta_0 - \ln \theta_1}{\ln \beta - \ln(1-\alpha)}$.