STA 413F2008 Assignment 4

The problems are practice for the Test 2, and are not to be handed in. Please start by reviewing your lecture notes and reading Section 4.3. You may skim 4.3.1 and 4.3.2. We will do 4.3.2 (the delta method) a somewhat different way in lecture. The main point of Section 4.3.3 is that convergence of moment-generating functions implies convergence in distribution. The main example is the Central Limit Theorem, but you are not responsible for the proof.

When answering the questions, it is wise to use the formula sheet in preference to theorems from the textbook. When you use something from the formula sheet, please cite it explicitly, for example saying "By the Weak Law of Large Numbers ...," or "Slutsky (a) for convergence in distribution implies ...," or "From the formula sheet, $X_n \stackrel{P}{\rightarrow} X \Rightarrow X_n \stackrel{d}{\rightarrow} X$." See lecture notes for examples.

1. What is the moment-generating function of a degenerate random variable T with $P\{T = \theta\} = 1$?

2. Do Exercise 4.3.1 three ways:

   (a) Give one-line "proof" using two results from the formula sheet.

   (b) Use the definition of convergence in distribution. Hint: Do it separately for $x < \mu$ and $x > \mu$.

   (c) Use limiting moment-generating functions. You don't need to prove the distribution of $\overline{X}_n$.

3. Prove $\lim_{n\rightarrow\infty} \left(1 + \frac{x}{n}\right)^n = e^x$. Hint: Use L'Hôpital's rule.

4. Let $X_1, \ldots, X_n$ be independent and identically distributed random variables from a distribution with mean μ and moment-generating function $M(t)$. Use limiting moment-generating functions to prove the Weak Law of Large Numbers. Hint: Use L'Hôpital's rule, and recall that $M'(0) = \mu$.

5. Do Exercise 4.3.2. Use the definition.

6. Let $X_1, \ldots, X_n$ be a random sample from a distribution with a density that is uniform on the interval from zero to θ, and let Y_n denote the maximum. We will investigate the limiting behaviour of $T_n = n(1 - \frac{Y_n}{\theta})$.

   (a) What is the support of Y_n?

   (b) What is the support of T_n?

   (c) If T_n has a limiting distribution, what does the support of that limiting distribution have to be?

   (d) Now write the cumulative distribution function of T_n as $Pr\{n(1 - \frac{Y_n}{\theta}) \leq t\}$, simplify, and take the limit as $n \rightarrow \infty$. Use Problem 3.

   (e) If you recognize the result, great; what is the limiting distribution called? otherwise, differentiate it to obtain a familar density.

7. Do Exercise 4.3.3. Hints: Assume the cumulative distribution function F is strictly increasing and therefore has a unique inverse. Remember, Z_n must always be non-negative.

8. Do Exercise 4.3.5 using the definition.

9. Do Exercise 4.3.6. Hints: You may use the distribution of Z_n without proof. Standardize. Recall that for any cumulative distribution function, $\lim_{x\to\infty} F(x) = 1$ and $\lim_{x\to-\infty} F(x) = 0$.

10. Do Exercise 4.3.7. Hint: Limiting moment-generating functions and Problem 3.

11. Do Exercise 4.3.8. Hint: Fearlessly apply L'Hôpital's rule.

12. Do Exercise 4.3.9. Hint: For this problem and the next one, it is helpful to recall that convergence in distribution refers to convergence of cumulative distribution functions, and not of the random variables themselves. Thus, one may freely replace a random variable with a more convenient one that has the same distribution. In this case, let $W_1, \ldots, W_n$ be a random sample from a Chi-square distribution with $\nu = 1$. Using moment-generating functions if necessary (this was a homework problem in Assignment One), verify that $\sum_{i=1}^n W_i$ has the same distribution as X. Now you can use the Central Limit Theorem.

13. Do Exercise 4.3.11. Hint: The sum of independent Poissons is ...?

14. Let $X : \mathcal{C} \to \mathbb{R}$ be a random variable. Prove that

$$\frac{X}{n} \stackrel{a.s.}{\to} 0.$$

This is surprisingly fast.

15. Let $X_1, \ldots, X_n$ be independent and identically distributed random variables with mean μ and variance σ^2. Prove that S_n^2 is consistent for σ^2. When you use something from the formula sheet, say so.

16. Let $X_1, \ldots, X_n$ be a random sample from a Gamma distribution with parameters $\alpha > 0$ and $\beta > 0$. Is the estimator

$$\widehat{\alpha}_n = \frac{\overline{X}_n^2}{S_n^2}$$

consistent for α? Answer Yes or No and prove your answer. When you use something from the formula sheet, say so.

17. Let $Y_i = \beta X_i + \epsilon_i$, for $i = 1, \ldots, n$, where

    $X_1, \ldots, X_n$ are independent and identically distributed random variables with mean μ and variance $\sigma_x^2 > 0$.

    $\epsilon_1, \ldots, \epsilon_n$ are independent and identically distributed random variables with mean zero and variance σ_ϵ^2.

    The Xs and ϵs are independent of each other.

    The parameters β, μ, σ_x^2 and σ_ϵ^2 are unknown.

Notice that this is different from the case where the x_i values are fixed constants.

(a) Let

$$\widehat{\beta}_{1,n} = \frac{\sum_{i=1}^n X_i Y_i}{\sum_{i=1}^n X_i^2}.$$

i. Since $E(X_iY_i)$ is the same for $i = 1, \ldots, n$, we will just call it $E(XY)$. Write $E(XY)$ in terms of the model parameters. Show your work.

ii. Define $Z_i = X_iY_i$. How do you know that $\overline{Z}_n$ is consistent for $E(XY)$?

iii. Prove that $\widehat{\beta}_{1,n}$ is consistent for β. When you use something from the formula sheet, say so.

iv. Is $\widehat{\beta}_{1,n}$ also unbiased? Answer Yes or No and show your work. This is a bit challenging because the X_i values are random variables. Hint: Use double expectation. If it helps to assume that the Xs have a continuous distribution with density $f(x)$, go ahead.

(b) Let $\widehat{\beta}_{2,n} = \frac{\overline{Y}_n}{\overline{X}_n}$.

i. Prove that $\widehat{\beta}_{2,n}$ is consistent for β. When you use something from the formula sheet, say so. Careful! Do you need an additional assumption about the distribution of $X_1, \ldots, X_n$?

ii. Is $\widehat{\beta}_{2,n}$ also unbiased? Answer Yes or No and show your work. Again it will help to use double expectation. Please assume that $X_1, \ldots, X_n$ come from a continuous distribution. This will prevent $Pr\{\overline{X}_n = 0\} > 0$, which would spoil everything.

18. Let $f_{X_n}(x) = \frac{1}{4}I(x = 0) + \frac{1}{2}I(x = 1) + \frac{1}{4}I(x = \frac{n+1}{n})$.

(a) Is X_n discrete, or is it continuous?

(b) What is $F_{X_n}(x)$? You may write the answer as a case function, or you may write it using indicators.

(c) Let $g(x) = \lim_{n\to\infty} f_{X_n}(x)$. Consider the cases $x = 0$, $x = 1$ and x equals something else separately. Is $g(x)$ a probability distribution?

(d) Let $G(x) = \lim_{n\to\infty} F_{X_n}(x)$. Your answer should apply to all x. Is $G(x)$ a cumulative distribution function?

(e) Let X be a Bernoulli random variable with $\theta = \frac{3}{4}$. Denote the cumulative distribution function of X by $F_X(x)$. At what points is $F_X(x)$ discontinuous? Does $G(x)$ equal $F_X(x)$ except possibly at those points? Does this mean $X_n \stackrel{d}{\to} X$? (Check the definition.)

(f) Do we have $\lim_{n\to\infty} E(X_n) = E(X)$? Answer Yes or No. Show your work.

(g) Do we have $\lim_{n\to\infty} Var(X_n) = Var(X)$? Answer Yes or No. Show your work.

19. Let the discrete random variable X_n have probability mass function $p_{X_n}(x) = \frac{1}{3}I(x = 0) + \frac{2}{3}\left(\frac{n-1}{n}\right)I(x = 1) + \frac{2}{3n}I(x = n)$.

(a) What is $F_{X_n}(x)$? You may write the answer as a case function, or you may write it using indicators.

(b) Let $p(x) = \lim_{n\to\infty} p_{X_n}(x)$. Consider the cases $x = 0$, $x = 1$ and x equals something else separately. Is $p(x)$ a probability distribution?

(c) Let $F(x) = \lim_{n\to\infty} F_{X_n}(x)$. Your answer should apply to all x.

i. What is $F(x)$? You may write the answer as a case function, or you may write it using indicators.

ii. Is $F(x)$ a cumulative distribution function? Does it correspond to $p(x)$?

iii. We seem to have $X_n \xrightarrow{d} X$, where X is Bernoulli again. What is the parameter θ?

(d) Do we have $\lim_{n\to\infty} E(X_n) = E(X)$? Answer Yes or No. Show your work.

(e) Do we have $\lim_{n\to\infty} Var(X_n) = Var(X)$? Answer Yes or No. Show your work.