STA 413F2008 Assignment 11

The following questions are practice for the Final Examination. They are not to be handed in.

1. Let $X_1, \ldots, X_n$ be a random sample from a distribution with density $f(x;\theta_0)$.

   (a) The random variable
   $$\frac{1}{n}\sum_{i=1}^{n} \frac{\partial}{\partial\theta} \ln f(X_i;\theta)$$
   converges in probability to something. What is the target? How do you know?

   (b) The random variable
   $$-\frac{1}{n}\sum_{i=1}^{n} \frac{\partial^2}{\partial\theta^2} \ln f(X_i;\theta)$$
   converges in probability to something. What is the target? How do you know?

   (c) Write down a Taylor expansion (two terms plus remainder) of the function $\ell'(\theta)$; expand about the true parameter θ_0.

   (d) Assuming the remainder term may be disregarded, find the limiting distribution of $\sqrt{n}(\widehat{\theta}_n - \theta_0)$. Cite facts from the formula sheet and Homework Assignment 10 as you use them.

   (e) Prove $\sqrt{nI(\widehat{\theta}_n)}\,(\widehat{\theta}_n - \theta_0) \stackrel{d}{\rightarrow} Z \sim N(0,1)$, citing facts from the formula sheet as you use them.

   (f) Derive a $(1-\alpha)100\%$ confidence interval for θ. Show your work.

2. Let $X_1, \ldots, X_n$ be a random sample from a distribution with density
   $$f(x;\theta) = (\theta+1)x^{\theta} I(0 < x < 1),$$
   where $\theta > 0$.

   (a) Verify
   $$\mu = \frac{\theta+1}{\theta+2} \quad \text{and} \quad \sigma^2 = \frac{\theta+1}{(\theta+3)(\theta+2)^2}.$$

   (b) Let
   $$T_n = \frac{2\overline{X}_n - 1}{1 - \overline{X}_n}.$$
   Is T_n a consistent estimator of θ? Answer Yes or No and prove your answer, citing facts from the formula sheet as you use them.

   (c) Use the delta method to show $\sqrt{n}(T_n - \theta_0) \stackrel{d}{\rightarrow} X$. What is the distribution of X? Give its mean and variance.

   (d) Obtain a formula for the MLE of θ. Show your work.

(e) Calculate the Fisher Information $I(\theta)$. Show your work.

(f) $\sqrt{n}(\widehat{\theta}_n - \theta) \xrightarrow{d} Y$. What is the distribution of Y? Give its mean and variance.

(g) Compare $Var(X)$ from Question 2c to $Var(Y)$ from Question 2f. Which is smaller? Prove your answer. Of course smaller is better.

(h) A random sample of size $n = 50$ yields $\sum_{i=1}^{n} X_i = 39.557$ and $\sum_{i=1}^{n} \ln(X_i) = -12.947$.

i. What is T_n? Your answer is a single number.

ii. What is $\widehat{\theta}_n$? Your answer is a single number.

iii. Give an approximate 95% confidence interval for θ based on T_n. Your answer is a pair of numbers.

iv. Give an approximate 95% confidence interval for θ based on $\widehat{\theta}_n$. Your answer is a pair of numbers.

3. Let $X_1, \ldots, X_n$ be a random sample from a geometric distribution with parameter θ.

(a) Let $T_n = 1/\overline{X}_n$. Is T_n a consistent estimator of θ? Answer Yes or No and prove your answer, citing facts from the formula sheet as you use them.

(b) Use the delta method to show $\sqrt{n}(T_n - \theta) \xrightarrow{d} X$. What is the distribution of X? Give its mean and variance.

(c) Obtain a formula for the MLE of θ. Show your work.

(d) Calculate the Fisher Information $I(\theta)$. Show your work.

(e) $\sqrt{n}(\widehat{\theta}_n - \theta) \xrightarrow{d} Y$. What is the distribution of Y? Give its mean and variance.

(f) Compare $Var(X)$ from Question 3b to $Var(Y)$ from Question 3e.

(g) A random sample of size $n = 100$ yields $\overline{X}_n = 1.44$.

i. What is T_n? Your answer is a single number.

ii. What is $\widehat{\theta}_n$? Your answer is a single number.

iii. Give an approximate 95% confidence interval for θ based on T_n. Your answer is a pair of numbers.

iv. Give an approximate 95% confidence interval for θ based on $\widehat{\theta}_n$. Your answer is a pair of numbers.