STA 413F2008 Assignment 10

Please read Section 6.2; notice Example 6.2.4. The following questions are practice for the Final Examination. They are not to be handed in.

1. Let the discrete random variable X have probability mass function $p(x;\theta)$, and assume the regularity condition

$$\frac{\partial^2}{\partial\theta^2}\sum_x p(x;\theta) = \sum_x \frac{\partial^2}{\partial\theta^2}p(x;\theta).$$

   (a) Is the regularity condition satisfied by the Bernoulli distribution? Answer Yes or No and justify your answer.

   (b) Let the *score* random variable $S = \frac{\partial}{\partial\theta}\ln p(X;\theta)$. Show $E(S) = 0$.

   (c) Show $Var(S) = I(\theta)$, where

$$I(\theta) = -E\left(\frac{\partial^2}{\partial\theta^2}\ln p(X;\theta)\right).$$

   (d) Let $X_1, \ldots, X_n$ be a random sample from a discrete distribution with probability mass function $p(x;\theta)$, let $Y = u(X_1, \ldots, X_n)$ have expected value $E(Y) = k(\theta)$, and let $Z = \sum_{i=1}^n S_i$. Prove $k'(\theta) = E(YZ)$.

   (e) What is $E(Z)$?

   (f) What is $Cov(Y, Z)$?

   (g) What is $Corr(Y, Z)$?

   (h) Using the fact that the absolute value of a correlation cannot be greater than one, establish the Cramér-Rao inequality of Theorem 6.2.1.

   (i) Give a lower bound for the variance of any unbiased estimator of θ, based on a rndom sample from this distribution.

2. Let $X_1, \ldots, X_n$ be a random sample from a Poisson distribution with parameter λ. Find the MLE of λ. Is it efficient? Answer Yes or No and prove your answer.

3. Let $X_1, \ldots, X_n$ be a random sample from a Binomial distribution with parameters 4 and θ. Find the MLE of θ. Is it efficient? Answer Yes or No and prove your answer.

4. Let $X_1, \ldots, X_n$ be a random sample from a distribution with density $f(x;\theta) = \theta x^{\theta-1} I(0 < x < 1)$, where $\theta > 0$.

   (a) Calculate the Fisher information $I(\theta)$.

   (b) What is the lowest possible variance for an unbiased estimator of θ, based on a random sample of size n? Your answer is a function of n and θ.

   (c) Find the Maximum Likelihood Estimator of θ.

   (d) Find the density of $Y = -\ln(X)$, where X has the density of this problem.

   (e) Show that the MLE has a well-known distribution. What are the parameters?

   (f) Show

   $$Var(\widehat{\theta}_n) = \frac{\theta^2 n^2}{(n-1)^2(n-2)}.$$

   (g) Which is greater, $Var(\widehat{\theta}_n)$ or the lower bound $\frac{1}{nI(\theta)}$?

   (h) Calculate

   $$\lim_{n\to\infty} \frac{Var(\widehat{\theta}_n)}{\frac{1}{nI(\theta)}}.$$

   What does this tell you?

5. Do Exercises 6.2.1, 6.2.2, 6.2.7 and 6.2.8.