STA 347F2003 Quiz 8

1. Let $X_0, X_1, \ldots$ be a stationary Markov chain with transition probabilities $P_{i,i} = \alpha$, $P_{i,i+1} = 1 - \alpha$, and zero otherwise, where $0 < \alpha < 1$.

   (a) (2 Points) Is this Markov chain irreducible? Answer Yes or No.

   (b) (3 Points) What is the period of state 3? Why?

   (c) (5 Points) What is $f_{3,3}^{(n)}$?

   (d) (5 Points) What is $f_{3,3}$?

   (e) (2 Points) Is state 3 recurrent? Answer Yes or No.

   (f) (5 Points) What is $P_{00}^{(n)}$?

   (g) (5 Points) Use your answer to the preceding item to show recurrence or transience.

   (h) (3 Points) Does Theorem 4.2 apply? Answer Yes or No and say why.

   (i) (10 Points) Try to find the stationary distribution anyway. That is, find the solution to $\boldsymbol{\pi} = \boldsymbol{\pi}\mathbf{P}$.

   (j) (5 Points) Is the vector $\boldsymbol{\pi}$ you derived in the last question a probability distribution? Answer Yes or No, and say why.

   (k) (15 Points) Is your π_0 equal to $\lim_{n\to\infty} Pr\{X_n = 0\}$? (By "your" π_0 I mean element zero of the vector $\boldsymbol{\pi}$ you derived). Answer Yes or No, and prove it. Hint: Start with $Pr\{X_n = 0\} = \sum_{k=0}^{\infty} Pr\{X_n = 0 | X_0 = k\} Pr\{X_0 = k\}$. You can proceed even though you do not know $Pr\{X_0 = k\}$.

2. (40 Points) Let $X_0, X_1, \ldots$ be a stationary Markov chain with transition matrix

	0	1	2	3	4	5
0	0	1	0	0	0	0
1	$\frac{1}{2}$	0	$\frac{1}{2}$	0	0	0
2	0	$\frac{1}{2}$	0	$\frac{1}{2}$	0	0
3	0	0	$\frac{1}{2}$	0	$\frac{1}{2}$	0
4	0	0	0	$\frac{1}{2}$	0	$\frac{1}{2}$
5	0	0	0	0	1	0

   You will recognize this as a random walk in which states zero and 5 reflect the process back into states 1 through 4; it is directly from a homework problem. Find $\lim_{n\to\infty} Pr\{X_n = 0\}$. Show your work and *circle your answer.*

Jerry's Answers to Quiz 8

1 a) No.

b) $d(3) = 1$ because $P_{33} > 0$

c) $f_{33}^{(1)} = \alpha$, $f_{33}^{(n)} = 0$ for $n > 1$

d) $f_{33} = \alpha + 0 + 0 + \cdots = \alpha$

e) No; it's transient because $f_{33} < 1$.

f) $P_{00}^{(n)} = \alpha^n$

g) $\sum_{n=1}^{\infty} \alpha^n = \frac{\alpha}{1-\alpha} < \infty$, transient

h) No, because the Markov chain is not irreducible.

i)

$$\underset{\sim}{P} = \begin{array}{c|ccccc} & 0 & 1 & 2 & 3 & 4 \\ \hline 0 & \alpha & 1-\alpha & & & \\ 1 & & \alpha & 1-\alpha & & \\ 2 & & & \alpha & 1-\alpha & \\ 3 & & & & \alpha & 1-\alpha \end{array}$$

$$\pi_0 = \alpha \pi_0 \Rightarrow \pi_0 = 0$$

$$\pi_1 = (1-\alpha)\pi_0 + \alpha\pi_1 = \alpha\pi_1 \Rightarrow \pi_1 = 0$$

$$\pi_2 = (1-\alpha)\pi_1 + \alpha\pi_2 = \alpha\pi_2 \Rightarrow \pi_2 = 0$$

$$\vdots$$

$$\pi_j = (1-\alpha)\pi_{j-1} + \alpha\pi_j = \alpha\pi_j \Rightarrow \pi_j = 0$$

So $\underset{\sim}{\pi} = (0, 0, \cdots)$

j) No, because $\sum_{k=0}^{\infty} \pi_k = 0 \neq 1$

1k) Yes. $\lim_{n\to\infty} \Pr\{X_n=0\} = \lim_{n\to\infty} \sum_{k=0}^{\infty} \Pr\{X_n=0 \mid X_0=k\} \Pr\{X_0=k\}$

Bec $\Pr\{X_n=0 \mid X_0=k\}=0$ for $k>0$

$\downarrow$

$= \lim_{n\to\infty} \Pr\{X_n=0 \mid X_0=0\} \Pr\{X_0=0\}$

$= \lim_{n\to\infty} P_{00}^{(n)} \Pr\{X_0=0\} = \Pr\{X_0=0\} \lim_{n\to\infty} \alpha^n = 0 = \pi_0 .$

2) $\pi_0 = \frac{1}{2}\pi_1 \implies \pi_1 = 2\pi_0$

$\pi_1 = \pi_0 + \frac{1}{2}\pi_2 \implies 2\pi_0 = \pi_0 + \frac{1}{2}\pi_2 \implies \pi_2 = 2\pi_0$

$\pi_2 = \frac{1}{2}\pi_1 + \frac{1}{2}\pi_3 \implies 2\pi_0 = \pi_0 + \frac{1}{2}\pi_3 \implies \pi_3 = 2\pi_0$

$\pi_3 = \frac{1}{2}\pi_2 + \frac{1}{2}\pi_4 \implies 2\pi_0 = \pi_0 + \frac{1}{2}\pi_4 \implies \pi_4 = 2\pi_0$

$\pi_4 = \frac{1}{2}\pi_3 + \pi_5 \implies 2\pi_0 = \pi_0 + \pi_5 \implies \pi_5 = \pi_0$

and

$$1 = \sum_{k=0}^{5} \pi_k = \pi_0(1+2+2+2+2+1) = 10\,\pi_0$$

$$\implies \pi_0 = \lim_{n\to\infty} \Pr\{X_n=0\} = \frac{1}{10}$$