STA 347F2003 Quiz 7

1. Let $X_0, X_1, \ldots$ be a stationary Markov chain with transition matrix

$$\begin{array}{c|cccc} & 0 & 1 & 2 & 3 \\ \hline 0 & \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3} \\ 1 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 2 & \frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} \\ 3 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \end{array}$$

   (a) (15 Points) Is this Markov chain regular? Answer Yes or No, and prove it.

   (b) (20 Points) Whether it is regular or not, find the limiting distribution. Justify your answer. There is a short way and a long way to do this part.

   (c) (5 Points) What fraction of the time, in the long run, does the process spend in state 1?

   (d) (10 Points) Every period that the process spends in state 0 incurs a cost of \$2. Every period that the process spends in state 1 incurs a cost of \$5. Every period that the process spends in state 2 incurs a cost of \$2. Every period that the process spends in state 3 incurs a cost of \$3. What is the long run average cost per period associated with this Markov chain?

2. (35 Points) Let $X_0, X_1, \ldots$ be a *regular* stationary Markov chain with state space $\{0, \ldots, N\}$. Prove $\lim_{n\to\infty} \mathbf{p}^{(n)} = \boldsymbol{\pi}$, or else disprove it by giving a simple counter-example, for example, one involving the repeated toss of a fair coin.

3. (15 Points) A regular stationary Markov chain with finite state space has transition probability matrix $\mathbf{P} = [P_{ij}]$ and limiting distribution $\boldsymbol{\pi} = [\pi_j]$. In the long run, what fraction of the *transitions* are from state 0 to state 1?

Q7Answ1

Jerry's Answers to Quiz 7

(1a)
$$\begin{bmatrix} + & + & 0 & + \\ 0 & + & + & + \\ + & 0 & + & + \\ + & + & + & 0 \end{bmatrix}\begin{bmatrix} + & + & 0 & + \\ 0 & + & + & + \\ + & 0 & + & + \\ + & + & + & 0 \end{bmatrix} = \begin{bmatrix} + & + & + & + \\ + & + & + & + \\ + & + & + & + \\ + & + & + & + \end{bmatrix}$$

YES

Or actual matrix multiplication showing $\underset{\sim}{P}^2$ has no zeros.

(b) The transition matrix is doubly stochastic, so
$\underset{\sim}{\pi} = \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right)$

(c) $\frac{1}{4}$ of the time

d) $\lim_{n\to\infty} E\left[c(X_n)\right] = \frac{1}{4}(2+5+2+3) = \frac{12}{4} = 3$

(2) Because the Markov chain is regular with a finite state space, we know $\lim_{n\to\infty} P_{ij}^{(n)} = \pi_j$ exists, with $\sum_{j=0}^{N} \pi_j = 1$. Now,

$$\lim_{n\to\infty} \Pr\{X_n = j\} = \lim_{n\to\infty} \sum_{k=0}^{N} \Pr\{X_n = j \mid X_0 = k\} \Pr\{X_0 = k\}$$

$$= \sum_{k=0}^{N} \Pr\{X_0 = k\} \lim_{n\to\infty} P_{ij}^{(n)} = \sum_{k=0}^{N} \Pr\{X_0 = k\}\, \pi_j$$

$$= \pi_j \sum_{k=0}^{N} \Pr\{X_0 = k\} = \pi_j \quad \blacksquare$$

It's okay to start with $\underset{\sim}{p}^{(n)} = \underset{\sim}{p}^{(0)} \underset{\sim}{P}^{n}$ instead of re-proving it.

Q7 Answer

(3) $\lim_{n\to\infty} Pr\{X_n = 0, X_{n+1} = 1\}$

$$= \lim_{n\to\infty} Pr\{X_{n+1} = 1 \mid X_n = 0\} Pr\{X_n = 0\}$$

$$= P_{01} \lim_{n\to\infty} Pr\{X_n = 0\} = P_{01}\,\pi_0$$