# STA 347F2003 Quiz 4

1. (30 Points) Let the discrete random variable $X$ take values $x_1, x_2, \ldots$, $Y$ take values $y_1, y_2, \ldots$, and $Z$ take values $z_1, z_2, \ldots$. Prove

$$Pr\{X = x_k | Y = y_j\} = \sum_{n=1}^{\infty} Pr\{X = x_k | Y = y_j, Z = z_n\} Pr\{Z = z_n | Y = y_j\},$$

or else disprove it by giving a simple counter-example.

2. (40 Points) Let $X_0, X_1, \ldots$ be a stationary Markov chain with transition probabilities satisfying $P_{ij} = a_j$ for all $i$ and $j$. Prove or disprove: $\mathbf{P}^n = \mathbf{P}$ for $n = 1, 2, \ldots$.

3. (30 Points) Let $X_0, X_1, \ldots$ be a stationary Markov chain with transition matrix

| | 0 | 1 |
|---|---|---|
| 0 | $a$ | $1-a$ |
| 1 | $b$ | $1-b$ |

,

and let $\mathbf{p}^{(3)} = [c, 1-c]$, where $a \neq b$ and $c$ is between $a$ and $b$. Find $Pr\{X_2 = 1\}$. Show your work.

# Jerry's Answers to Quiz 4

(1) $$RHS = \sum_{n=0}^{\infty} \Pr\{X = x_k \mid Y = y_j, Z = z_n\} \Pr\{Z = z_n \mid Y = y_j\}$$

$$= \sum_{n=0}^{\infty} \frac{\Pr\{X = x_k, Y = y_j, Z = z_n\}}{\Pr\{\cancel{Y = y_j, Z = z_n}\}} \frac{\cancel{\Pr\{Y = y_j, Z = z_n\}}}{\Pr\{Y = y_j\}}$$

$$= \frac{\Pr\{X = x_k, Y = y_j\}}{\Pr\{Y = y_j\}} = \Pr\{X = x_k \mid Y = y_j\} = LHS \quad \blacksquare$$

(2) <u>Proof by induction</u>

a) <u>Show for $n=1$</u> : $\underset{\sim}{P}^1 = \underset{\sim}{P}$ True.

b) <u>Assume for $n-1$</u> (Induction hypothesis) $\underset{\sim}{P}^{n-1} = \underset{\sim}{P}$

c) <u>Show for $n$</u>

$$\underset{\sim}{P}^n = \underset{\sim}{P}^{n-1} \underset{\sim}{P} = \underset{\sim}{P}\underset{\sim}{P} = \left[\sum_{k=0}^{\infty} P_{ik} P_{kj}\right]$$

$$= \left[\sum_{k=0}^{\infty} a_k a_j\right] = \left[a_j \underbrace{\sum_{k=0}^{\infty} a_k}_{1}\right] = [a_j] = [P_{ij}]$$

$$= \underset{\sim}{P}$$ <u>done</u>

3 $P_n\{X_3=0\} = P_n\{X_3=0 \mid X_2=0\}P_n\{X_2=0\} + P_n\{X_3=0 \mid X_2=1\}P_n\{X_2=1\}$

Letting $k = P_n\{X_2=1\}$, this is

$$c = a(1-k) + bk = a - ak + bk$$

$$\iff c-a = k(b-a) \iff k = \frac{c-a}{b-a} = P_n\{X_2=1\}$$