STA 347F2003 Quiz 10

1. (20 Points) Calls come in to an information hotline according to a Poisson process with rate $\lambda = 2$ per minute. What is the probability that more than 3 minutes elapse between the 7th and 8th calls? This one is fast if you know what to do.

2. Let $\{X(t) : t \geq 0\}$ be a Poisson process with rate λ, and let W_1 be the waiting time until the first event. Find the density of W_1 given that n events occurred in the interval $(0, t]$, by following these steps. Assume $n > 1$.

   (a) (15 Points) Find $Pr\{W_1 > w | X(t) = n\}$, where $0 < w < t$.

   (b) (5 Points) Give the cumulative distribution function $F_{W_1|X(t)=n}(w|X(t) = n)$.

   (c) (10 Points) Find the density $f_{W_1|X(t)=n}(w|X(t) = n)$.

   (d) (10 Points) For what values of w is the density function non-zero?

3. Let $\{X(t) : t \geq 0\}$ be a Poisson process with rate λ, and let W_2 be the waiting time until the *second* event. Find the density of W_2, by following these steps.

   (a) (20 Points) Find $Pr\{W_2 < w\}$, where $w > 0$.

   (b) (15 Points) Find the density $f_{W_2}(w)$. Remember the product rule $u'v + v'u$.

   (c) (5 Points) For what values of w is the density function non-zero?

STA 347F2003 Formula Sheet 2

If $0 < a < 1$ then $\sum_{k=j}^{\infty} a^k = \frac{a^j}{1-a}$.

$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{k=0}^{\infty} \frac{x^k}{k!}$.

$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$.

Binomial: $X \sim B(n, p)$ means $Pr\{X = k\} = \binom{n}{k} p^k (1-p)^{n-k}$ for $k = 0, , \ldots , n$. Note: $E[X] = np$, $V[X] = np(1-p)$.

Poisson: $X \sim P(\mu)$ means $Pr\{X = k\} = \frac{e^{-\mu}\mu^k}{k!}$ for $k = 0, 1, \ldots$. Note: $E[X] = V[X] = \mu$

Exponential: $X \sim \exp(\lambda)$ means $f_X(x) = \lambda e^{-\lambda x} \mathbf{1}\{x \geq 0\}$. Note $E[X] = 1/\lambda$, and $F_X(x) = (1 - e^{-\lambda x})\mathbf{1}\{x \geq 0\}$.

Gamma: $X \sim G(n, \lambda)$ means $f_X(x) = \frac{\lambda^n}{(n-1)!} e^{-\lambda x} x^{n-1} \mathbf{1}\{x \geq 0\}$. Note $E[X] = \frac{\lambda}{n}$. The sum of n independent exponential random variables is Gamma.

Q10ANSW1

Jerry's Answers to Quiz 10

① Sojurn times are exponential (λ), so

$$\Pr\{S_7 > 3\} = e^{-2 \cdot 3} = e^{-6}$$

② a) $$\Pr\{W_1 > w \mid X(t) = n\} = \frac{\Pr\{X(w) = 0,\ X(t) = n\}}{\Pr\{X(t) = n\}}$$

$$= \frac{\Pr\{X(w) = 0,\ X(t) - X(w) = n\}}{\Pr\{X(t) = n\}} = \frac{\Pr\{X(w) = 0\}\Pr\{X(t) - X(w) = n\}}{\Pr\{X(t) = n\}}$$

$$= \frac{\cancel{e^{-\lambda w}}\ \dfrac{\cancel{e^{-\lambda(t-w)}}[\lambda(t-w)]^n}{n!}}{\dfrac{\cancel{e^{-\lambda t}}(\lambda t)^n}{\cancel{n!}}} = \left(\frac{t-w}{t}\right)^n$$

$$= \left(1 - \frac{w}{t}\right)^n$$

b) $$F_{W_1 \mid X(t)=n}(w \mid X(t) = n) = 1 - \left(1 - \frac{w}{t}\right)^n$$

c) $$f_{W_1 \mid X(t)=n}(w_1 \mid X(t) = n) = (-1)\, n \left(1 - \frac{w}{t}\right)^{n-1}\left(-\frac{1}{t}\right)$$

$$= \frac{n}{t}\left(1 - \frac{w}{t}\right)^{n-1}$$

d) $0 < w < t$

Q10 Answ 2

③ a) $Pr\{W_2 \le w\} = Pr\{X(w) \ge 2\}$

$= 1 - Pr\{X(w) < 2\} = 1 - Pr\{X(w) = 0\} - Pr\{X(w) = 1\}$

$= 1 - e^{-\lambda w} - e^{-\lambda w}\lambda w = 1 - e^{-\lambda w}(1 + \lambda w)$

b) $f_{W_2}(w) = \frac{d}{dw}\left(1 - e^{-\lambda w}(1 + \lambda w)\right)$

$= (-1)\frac{d}{dw}\left[e^{-\lambda w}(1 + \lambda w)\right]$

$= (-1)\left[-\lambda e^{-\lambda w}(1 + \lambda w) + e^{-\lambda w}\lambda\right]$

$= (-1)\left[\cancel{-\lambda e^{-\lambda w}} - \lambda^2 w e^{-\lambda w} + \cancel{\lambda e^{-\lambda w}}\right]$

$= \lambda^2 w e^{-\lambda w} = \frac{\lambda^2}{(2-1)!} e^{-\lambda w} w^{2-1} \mathbb{1}\{w \ge 0\}$

c) $w \ge 0$