STA 347F2003 Assignment 4

Do this assignment in preparation for the quiz on Friday, Oct. 10th. It is not to be handed in.

1. Let the random variable

   - X take values $x_1, x_2, \ldots$
   - Y take values $y_1, y_2, \ldots$
   - Z take values $z_1, z_2, \ldots$

   This means, for example, that $Pr\{X = x_k\} > 0$ for $k = 1, 2, \ldots$, and that $\sum_{k=1}^{\infty} Pr\{X = x_k\} = 1$.

   (a) Prove $Pr\{X = x_k\} = \sum_{n=1}^{\infty} Pr\{X = x_k | Y = y_n\} Pr\{Y = y_n\}$, or else disprove it by giving a simple counter-example.

   (b) Prove $Pr\{X = x_k | Y = y_j\} = \sum_{n=1}^{\infty} Pr\{X = x_k | Y = y_j, Z = z_n\} Pr\{Z = z_n | Y = y_j\}$, or else disprove it by giving a simple counter-example.

2. Let $X_0, X_1, \ldots$ be a stationary Markov chain. Prove that $Pr\{X_3 = j | X_0 = i_0, X_1 = i\} = Pr\{X_3 = j | X_1 = i\}$, or else disprove it by giving a simple counter-example.

3. A stationary Markov chain has one-step transition matrix $\mathbf{P}$, $n-$step transition matrix $\mathbf{P}^n$, and vector of *unconditional* probabilities for X_n given by $\mathbf{p}^{(n)}$. Prove the following results, or else disprove one or both of them by giving simple counter-examples.

   (a) $\mathbf{p}^{(n)} = \mathbf{p}^{(0)} \mathbf{P}^n$.

   (b) $\mathbf{p}^{(n)} = \mathbf{p}^{(n-1)} \mathbf{P}$.

4. Let $X_0, X_1, \ldots$ be a stationary Markov chain. Prove or disprove that $Pr\{X_0 = i_0, \ldots, X_n = i_n\} = Pr\{X_1 = i_0, \ldots, X_{n+1} = i_n\}$ for $n = 1, 2, \ldots$.

5. Let $X_0, X_1, \ldots$ be a stationary Markov chain with transition matrix

	0	1
0	α	$1-\alpha$
1	$1-\alpha$	α

   ,

   and let $\mathbf{p}^{(0)} = [\frac{1}{2}, \frac{1}{2}]$. What is $\mathbf{p}^{(30)}$?

6. Let $X_0, X_1, \ldots$ be a sequence of *independent and identically distributed* discrete random variables, with $Pr\{X_n = k\} = a_k$ for $k = 1, 2, \ldots$ and $n = 0, 1, 2, \ldots$.

   (a) Is it a stationary Markov chain? Answer Yes or No and prove it.

   (b) If it is a stationary Markov chain, give its transition probability matrix.

7. Let $X_0, X_1, \ldots$ be a stationary Markov chain with transition matrix

	0	1	2	3
0	a_1	a_2	a_3	a_4
1	a_1	a_2	a_3	a_4
2	a_1	a_2	a_3	a_4
3	a_1	a_2	a_3	a_4

,

   (a) What is $\mathbf{P}^2$?

   (b) What is $\mathbf{P}^{20}$?

   (c) What is $\mathbf{p}^{(20)}$?

8. Prove or disprove: For a stationary Markov chain with $P_{ij} = a_j$ for all i and j, $\mathbf{P}^n = \mathbf{P}$ for $n = 1, 2, \ldots$.

9. Let $X_0, X_1, \ldots$ be a stationary Markov chain with transition matrix

	0	1
0	0.7	0.3
1	0.4	0.6

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   and let $\mathbf{p}^{(1)} = [\frac{1}{2}, \frac{1}{2}]$.

   (a) What is $\mathbf{p}^{(2)}$?

   (b) What is $\mathbf{p}^{(0)}$? Hint: write the Law of Total Probability for $Pr\{X_1 = 0\}$.

10. Starting from $\mathbf{p}^{(n)} = \mathbf{p}^{(0)}\mathbf{P}^n$, is it always possible to solve for $\mathbf{p}^{(0)}$ in terms of $\mathbf{p}^{(n)}$ and $\mathbf{P}$? Answer Yes or No. Hint: Think of Problem 7

11. We will now see that a stationary Markov chain is *reversible* – that is, it is still a Markov chain when you go backwards. Let $X_0, X_1, \ldots$ be a stationary Markov chain. What we really need to show is that for all $m < n$, $Pr\{X_m = j | X_n = i_n, X_{n-1} = i_{n-1}, \ldots, X_{m+1} = i\} = Pr\{X_m = j | X_{m+1} = i\}$, but to make the notation simpler, just show $Pr\{X_0 = j | X_5 = i_5, X_4 = i_4, X_3 = i_3, X_2 = i_2, X_1 = i\} = Pr\{X_0 = j | X_1 = i\}$. All you have to do is use the definition of conditional probability, the multiplication rule, and the usual forward Markov property. The general proof has the same structure.

12. Backwards transition probabilities are easy to obtain. Letting $m < n$, show that $Pr\{X_m = j | X_n = i\} = P_{ji}^{(n-m)} \frac{p_j^{(m)}}{p_i^{(n)}}$. Why does this formula suggest that the backwards Markov chain might not be stationary?

13. Let a stationary Markov chain have the transition matrix given in Problem 9, and as before, let $\mathbf{p}^{(1)} = [\frac{1}{2}, \frac{1}{2}]$. You have already calculated $\mathbf{p}^{(0)}$ and $\mathbf{p}^{(2)}$. What is $Pr\{X_0 = 0 | X_1 = 0\}$? What is $Pr\{X_1 = 0 | X_2 = 0\}$? What do you conclude?