STA 347F2003 Assignment 10

Do this assignment in preparation for the quiz on Friday, Nov. 28th. It is not to be handed in.

Read Sections 1 and 3 in Chapter 5. Then do the following questions. You will be given a copy of Formula Sheet 2 (see link on the class home page) for the quiz.

1. Let X have an exponential distribution with parameter λ. Also, let $t > 0$ and $s > 0$. Find
   (a) $Pr\{X > t\}$ (It's okay to use the formula sheet).
   (b) $Pr\{X > s + t | X > s\}$. Show your work.
2. Let $\{X(t) : t \geq 0\}$ be a Poisson process with rate λ; also, let $0 < u < t$ and $0 \leq k \leq n$. Find $Pr\{X(u) = k | X(t) = n\}$.
3. Let $0 < u < t$ and $0 \leq k \leq n$. Let $U_1, \ldots, U_n$ be *independent* continuous random variables with a uniform distribution on the interval $(0, t)$. What is the probability that k of the uniform random variables fall into the interval $(0, u)$?
4. Let $\{X(t) : t \geq 0\}$ be a Poisson process with rate λ, and let W_n be the waiting time until the nth event. Derive the probability density function of W_n. The distribution has a name; what is it?
5. Let $\{X(t) : t \geq 0\}$ be a *nonhomogeneous* Poisson process with rate function $\lambda(t) = t^2$. What is the probability of k events in $(0, 3]$?
6. Let $\{X(t) : t \geq 0\}$ be a *nonhomogeneous* Poisson process with rate function $\lambda(t) = \frac{1}{t}$. What is the probability of k events in $(a, b]$, where $0 < a < b$?
7. Let $\{X(t) : t \geq 0\}$ be a *nonhomogeneous* Poisson process with rate function $\lambda(t) = \frac{1}{(1+t)^2}$, and let $a > 0$. What is the probability of
   (a) k events in $(0, a]$?
   (b) k events in $(0, \infty)$?
8. Let $\{X(t) : t \geq 0\}$ be a Poisson process with rate λ.
   (a) Let T be the time at which the first event occurred. Derive its density.
   (b) Given that exactly one event occurred up until time t, what is the density of the time at which it occurred?
9. The set A of real numbers is defined by $A = (a, a + s) \cup (b, b + t - s)$, where $0 < a < a + s < b < b + t - s$, so that the total length of A is equal to t. Let $\{X(t) : t \geq 0\}$ be a Poisson process with rate λ. What is the probability that exactly k events occur in the set A? So we see that what matters is the length of the set, not whether it is split up.
10. Let $\{X(t) : t \geq 0\}$ be a Poisson process with rate λ. Given that 8 events occurred in the interval (0,10], what is the probability that 4 of them occurred in the interval (2,5]?

11. Let $\{N(a,b] : 0 < a < b\}$ be a Poisson process with rate λ. What is $Pr\{N(2,4] = 2|N(3,6] = 5\}$? Hint: add up three probabilities.

12. Starting on page 294, do Exercises 3.1 through 3.4, 3.5, 3.6[1], 3.8, 3.9

13. Starting on page 295, do Problems 3.1[2], 3.4[3], 3.5[4], 3.6[5], 3.7[6], 3.8[7]

14. Let $\{Y(t) : t \geq 0\}$ be a Poisson process with rate λ. Define $X_0 = 0$, and $X_n = Y(n)$ for $n = 1, 2, \ldots$. Is $\{X_0, X_1, \ldots\}$ a Markov chain? Is it stationary? Answer Yes or No to each question. To make your work easier, just check the case $n = 5$; the general case is similar.

---

[1] Maximum of n waiting times for first event. The book just gives the cumulative distribution function, but please get the density too.

[2] To get the probability involving "0 or 1," you need to add. Before differentiating your answer, write $Pr\{W_1 > w_1, W_2 > w_2\}$ as a double integral and use the Fundamental Theorem of Calculus to verify that you really do get the joint density by differentiating it twice. Then, when you do differentiate with respect to w_1 and w_2, note that one order of differentiation is much easier than the other.

[3] It will help if you sketch the region over which you are integrating.

[4] Check your answer by seeing if it adds up to one.

[5] $N(t)$ is a step function with jumps at the waiting times; $T = W_Q$. Sketch this function (you can see what's going on from an example with $Q = 4$), and obtain the integral as a sum of areas of rectangles. But the (random) lengths of the bases are the sojourn times . . . . And you will need the formula $\sum_{j=1}^{n} j = \frac{n(n+1)}{2}$.

[6] Letting $T_1, T_2, \ldots$ be the exponential failure times, you want $\min_n\{n : Pr\{\sum_{k=1}^{n} T_k > 1\} > .98\}$. Use Exercise 3.9 to get the Gamma probabilities you need. Make a table with two columns: n, and $Pr\{W_n > 1\}$. Note that $E[T_1] = 1/\lambda$. The answer is 5.

[7] Just get the (conditional) cumulative distribution function $Pr\{W_r \leq u|X(t) = n\}$. It's a sum of binomial probabilities. If you differentiate this sum to get a density, it's a big mess. The mess simplifies, but the simplification is unusually hard. Forget it.