University of Toronto
Faculty of Arts and Science
December Examinations 2000
STA 347F
Duration - 3 hours
Aids allowed: Textbook

1. (3 Points) Let X and Y be *independent* continuous random variables. Either

   - Prove $E[g(X)|Y = y] = E[g(X)]$, or
   - Show that the formula is not true in general by providing an example of two continuous independent random variables and a function $g(x)$ where $E[g(X)|Y = y] \neq E[g(X)]$.

   You must either prove or disprove the formula. If you do both, you will get zero marks even if one of your answers is correct. **Clearly indicate where you use independence.**

2. (5 Points) I roll a fair six-sided die and observe the number N on the uppermost face. I then toss a fair coin N times, and observe X, the total number of heads to appear. What is $E[X]$?

3. Let X be a discrete random variable and let Y be continuous. Assume $f_{Y|X}(y|x) = \frac{d}{dy}Pr\{Y \leq y|X = x\}$ exists, and define $p_{X|Y}(x|y)$ as $\lim_{\triangle y \downarrow 0} Pr(X = x \,|\, y < Y \leq y + \triangle y)$.

   (a) (8 Points) Prove $p_{X|Y}(x|y) = \frac{f_{Y|X}(y|x)p_X(x)}{f_Y(y)}$

   (b) (4 Points) Use the expression in part (a) to derive a double expectation formula for $E[g(X)]$ when X is discrete and Y is continuous. You may use the fact that $f_{Y|X}(y|x)$ integrates to one, without proof.

4. (6 Points) Suppose X has a Poisson distribution with mean M, and M is exponential with parameter $\lambda > 0$. What is the marginal pmf $p_X(x)$? The support is worth half marks.

5. (8 Points) For a Poisson process $X(t)$ with rate $\lambda > 0$, the number of events in any interval of length t has a Poisson distribution with mean λt. Use this fact to show that show that $Pr\{X(h) \geq 1\} = \lambda h + o(h)$ as $h \downarrow 0$.

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6. (7 Points) Let $X_0, X_1, \ldots$ be a regular stationary Markov chain with state space $\{0, \ldots, N\}$, so that $\lim_{n\to\infty} P_{i,j}^{(n)} = \pi_j$ for $j = 0, \ldots, N$, and $\sum_{j=0}^{N} \pi_j = 1$. Let the $1 \times (N+1)$ vector $\boldsymbol{\alpha} = (\alpha_0, \ldots, \alpha_N)$ satisfy $\sum_{j=0}^{N} \alpha_j = 1$ and $\boldsymbol{\alpha} = \boldsymbol{\alpha}\boldsymbol{P}^{(n)}$ for $n = 1, 2, \ldots$. Show $\boldsymbol{\alpha} = \boldsymbol{\pi}$, where $\boldsymbol{\pi} = (\pi_0, \ldots, \pi_N)$.

7. Let $X_0, X_1, \ldots$ be a stationary Markov chain with transition matrix

	0	1	2
0	0.1	0.1	0.8
1	0.2	0.2	0.6
2	0.3	0.3	0.4

   (a) (4 Points) What is $Pr(X_{27} = 2 | X_{25} = 0)$? Show your work, what there is of it.

   (b) (4 Points) Suppose $Pr(X_2 = 0) = 0.1$, $Pr(X_2 = 1) = 0.4$ and $Pr(X_2 = 2) = 0.5$. What is $Pr(X_2 = 0, X_3 = 1, X_4 = 2)$? Show your work.

   (c) (4 Points) Again, suppose $Pr(X_2 = 0) = 0.1$, $Pr(X_2 = 1) = 0.4$ and $Pr(X_2 = 2) = 0.5$. What is $Pr(X_3 = 1)$? Show some calcuations.

8. Two white and two black balls are distributed in two jars in such a way that each jar contains two balls. At each step, we simultaneously draw one ball at random from each jar, placing the ball drawn from each jar into the other jar. Let X_n denote the number of white balls in urn one, for $n = 1, \ldots$.

   (a) (4 Points) Give the transition probability matrix for this Markov chain.

   (b) (4 Points) Does the limiting probability distribution of X_n exist? Answer Yes or No, and justify your answer. No marks for a correct answer without justification.

   (c) (8 Points) Calculate the limiting probability distribution of X_n, *if it exists*. Otherwise, receive full marks for writing "Limiting probabilities do not exist".

9. (10 Points) An explorer wakes up in a maze with two chambers. Chamber one has four doors. One door leads to freedom. Doors two and three are the two ends of a loop that leads from Chamber One back to Chamber One again. The last door leads to Chamber Two. Chamber Two has three doors. One leads back to Chamber One, one leads to freedom, and the last leads to a pit of fire. Assume that at each point, the explorer is equally likely to choose each door (even if she has just entered by it). Starting in Chamber One, what is the probability that the explorer will eventually be free, and not burn in the pit of fire? Show your work.

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10. (8 Points) For a Poisson process with rate λ, derive the density of the time until the first event occurs. Do *not* derive the waiting time until the nth event and then set $n = 1$, the way the book does. And remember the support!

11. (8 Points) Let $X(t)$ be a *nonhomogeneous* Poisson process with rate $\lambda(t)$. For $0 < r < s$ and $0 \leq k \leq n$, show that $Pr\{X(r) = k | X(s) = n\}$ has a binomial form. Hint: do not be confused by the fact that the parameter p of the binomial is a ratio of integrals.

12. (5 Points) Pimples occur on a person's body according to a homogeneous Poisson process with rate $\lambda = \frac{1}{3}$ per square centimeter. Given that exactly one pimple exists in a square 2 cm by 2 cm area, what is the probability that it is in the lower right quarter? Show your work, using independent increments.

Total marks = 100 points