Probability Review

Hour test in tutorial Fri Jan 18: Tables 6.1 & 6.2 will be provided.

First read chapter 1. Then do these probs starting on p. 16: 6,7,14,21,26,31. Numbers 23 and 35 are classic problems; they will not be on the test, but take a look at them anyhow. Next,

1. Let S be a sample space and A & B be events in S with $P(B)>0$. Use the axioms of probability on p. 4 to prove that conditional probabilities are indeed probabilities, by completing i-iii below.

   i) Show that $0 \le P(A|B) \le 1$.

   ii) Show that $P(B|B)=1$.

   iii) Let $E_1, E_2, \ldots$ be a sequence of mutually exclusive events.

   Show that $P([\bigcup_{n=1}^{\infty} E_n]|B)=\sum_{n=1}^{\infty} P(E_n|B)$.

2. Let A be an event in the sample space S, and let $S=\bigcup_{k=1}^{n} E_k$ with $P(E_k)>0$ for all k and $E_iE_j=\varnothing$ for $i\neq j$. Prove the law of total probability, i.e. $P(A)=\sum_{k=1}^{n} P(A|E_k)P(E_k)$.

Now read Chapter 2,and do these probs starting on p. 75: 8, 14, 20(see prob 17), 23, 31, 35. Then do the following:

3. Let the random variable X have density $f_X(x)=2x$ for $0 \le x \le 1$ and zero elsewhere. Let $F_X(x)$ denote the distribution function of X. What is $F_X(2)$? Answ: 1

4. Let the random variable X have density

$$f_X(x)= \frac{1}{\sqrt{2\pi}} \exp\{4x-\tfrac{1}{2}x^2-8\}.$$ What is $P(X \ge 4)$? Answ: 1/2

5. Let X be exponentially distributed with parameter λ (see definition on p. 34). Show that $P(X \ge s+t \mid X>t)=P(X \ge s)$; that is, the exponential distribution has "no memory".

6. Let X be geometrically distributed with parameter p (see definition on p. 29). Show that $P(X \ge n+r \mid X>r)=P(X \ge n)$; that is, the geometric distribution has "no memory".

Now do problems 37, 47 and 48, starting on p. 79.

7. Let X be a gamma random variable with parameters n and λ. What is $E[X^k]$? Answ: $\dfrac{(n+k-1)!}{\lambda^k (n-1)!}$.

8. Let $X_1, \ldots, X_n$ be a collection of independent normal random variables with mean μ and variance σ^2. Show that the sample mean $\overline{X}$ has a normal distribution with mean μ and variance σ^2/n.

9. Suppose the number of males arriving per hour at a store is a Poisson random variable with parameter λ_1, and the number of females arriving per hour is a Poisson random variable with parameter λ_2. Assuming that these two random variables are independent, show that the total number of people arriving per hour has a Poisson distribution with parameter $\lambda_1+\lambda_2$.

10. Let $X_1, \ldots, X_n$ be a collection of independent Bernoulli(p) random variables. Show that $Y=\sum_{i=1}^{n} X_i$ has a Binomial distribution with parameters n and p.

Now read pp. 84-93, and do the following:

11. Let the continuous random variables X and Y have joint density $f_{XY}(x,y) = c\, e^{-(x+y)}$ for $0 \le y \le x$, and zero elsewhere.

a) What is c? Answ: 2

b) What is the marginal density of X?

Answ: $f_X(x)=2e^{-x}(1-e^{-x})$ for $x \ge 0$, and zero otherwise.

c) What is the conditional density of Y given X=x?

Answ: $e^{-y}/(1-e^{-x})$ for $0 \le y \le x$, and zero otherwise.

d) What is $E[Y \mid X=12]$?

Answ: $1 - 12e^{-12}/(1-e^{-12})=.99992627$.

e) What is $\lim_{x\to\infty} E[Y \mid X=x]$? Answ: 1

f) Are X and Y independent? Answ: No (why?)

12. Let the discrete random variables X and Y have joint probability mass function $f_{XY}(x,y)=(x+y)/36$ for $x=1,2,3$ and $y=1,2,3$; zero otherwise. What is $E[X|Y=2]$? Answ: 13/6.

13. Let a and b be constants, and let X and Y be discrete random variables. Show that $E[aX+b|Y]=aE[X|Y]+b$.

Now do probs 9-14 starting on p. 128. Finally, ...

14. Use the definition $\frac{d}{dx}g(x) = \lim_{h\to 0}\frac{1}{h}[g(x+h)-g(x)]$ to show that

a) $\frac{d}{dx}x^2 = 2x$.

b) $\frac{d}{dx}[a\,g(x)] = a\frac{d}{dx}g(x)$

c) The derivative of a constant is zero.