Erindale College - University of Toronto

Faculty of Arts and Science

April-May Examinations 1991

STA 347S

Duration - 3 hours

Aids allowed: Calculator

Name (Please print) ________________________

Signature _________________________________

Student Number ___________________________

1. Two white and two black balls are distributed in two urns in such a way that each contains two balls. At each step, we simultaneously draw one ball at random from each urn, placing the ball drawn from each urn into the other urn. Let X_n denote the number of white balls in urn one, for $n=1, 2, \ldots$.

a) (4 pts) Give the transition probability matrix for this Markov chain.

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1b) (4 pts) Calculate the limiting probability distribution of X_n, **IF IT EXISTS**. Otherwise, receive full marks for writing "Limiting probabilities do not exist".

2. Here is the matrix of transition probabilities for a Markov Chain.

$$\begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0.0 & 0.5 & 0.0 & 0.0 & 0.5 \\ 0.5 & 0.5 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.3 & 0.7 & 0.0 \\ 0.0 & 0.9 & 0.0 & 0.1 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 1.0 \end{bmatrix} \end{array}$$

a) (4 pts) Give the classes, and label each class as transient or recurrent.

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b) (2 pts) What is $P\{X_{102}=1 | X_{100}=3\}$?

c) (2 pts) Assuming that $P\{X_0=i\}=1/5$ for i=0, 1, ..., 4, What is $P\{X_1=4\}$?

d) (2 pts) The limiting probability distribution of X_n does exist for this problem. What is it? (Hint: THINK, don't calculate).

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3. (10 pts) The collection of integer valued random variables $\{X_n: n=0,1,...\}$ is said to be a Markov Chain if $P_{ij} = P\{X_{n+1}=j \mid X_n=i\} = P\{X_{n+1}=j \mid X_n=i, X_{n-1}=i_{n-1}, ..., X_0=i_0\}$ for all n=1, 2, ... That is, the future depends on the past only through the present. Using <u>only</u> this definition and the rules of conditional probability, show that $P\{X_{n+2}=j \mid X_n=i\} = \sum_k P_{ik} P_{kj}$

Warning: You are being asked to prove a special case of the Chapman-Kolmogorov equations. If you try to use those equations or anything that follows from them to solve this problem, you will receive <u>no credit at all</u>. Just use the definition above and things you know about conditional probability.

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4. (4 pts) Determine whether the function f(x)=log(1+x) is o(h). Show your work.

5. In a large city, babies are born at a Poisson rate of two per hour. Fifty percent are boys and fifty percent are girls.

a) (1 pt) What is the expected number of babies born during a 24 hour day?

b) (1 pt) What is the probability that exactly three girls are born during a six hour period?

c) (1 pt) What is the average time between births?

d) (1 pt) Starting at midnight, what is the expected waiting time until the tenth boy is born?

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e) (2 pts) A baby has just been born. What is the probability that it will be more than 5 hours before the next baby is born?

f) (2 pts) Starting at 4:00 pm, what is the probability that 3 girls will be born before 2 boys?

g) (2 pts) Given that 3 babies were born during the past hour, what is the probability that at least one of them was born during the first 15 minutes?

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6. (5 pts) Given that for a Poisson process, the number of events in any interval of length t has a Poisson distribution with mean λt, show that $P\{N(h)=1\} = \lambda h + o(h)$.

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7. (4 pts) A tavern in a small town opens at noon and closes at midnight. Let us say that t=0 represents noon. From noon until 6 pm, the Poisson rate at which customers arrive increases linearly from zero per hour at t=0 to 6 per hour at 6 pm. Then the arrival rate remains constant at 6 per hour until 10 pm. From 10 until midnight, the arrival rate is $6e^{-6(t-10)}$. Then at midnight everyone is thrown out and the arrival rate is zero until the next day at noon (reset the timer to t=0).

If no one who enters the tavern ever leaves voluntarily, what is the expected number of customers who are thrown out at midnight?

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8. (4 pts) An insurance company pays out claims on its life insurance policies according to a Poisson process having rate $\lambda=5$ per week. If the amount of money paid on each claim is exponentially distributed with mean $1000, then what is the probability that the company does not have to pay any money during a two-week time period? Hint: There is a quick ONE LINE solution to this problem.

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9. (5 pts) Let $\{N(t), t \geq 0\}$ be a nonhomogeneous Poisson process with intensity function $\lambda(s)$. Find the DENSITY $f_{T_1}(t)$ of T_1, the arrival time of the first event. Hint: it's not the ordinary exponential density.

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10. Consider two machines that are maintained by a single repair person. Machine 1 functions for an exponential time with rate μ_1 before breaking down, and machine 2 functions for an exponential time with rate μ_2 before breaking down; repair times are exponential with rate μ regardless of which machine is being repaired. All the exponential random variables are independent. <u>If the two machines are down at once, the repair person must fix machine 1 before starting on machine 2.</u> Let $X(t)=0$ if neither machine is down, $X(t)=1$ if machine 1 is down, $X(t)=2$ if machine 2 is down, and $X(t)=3$ if both machines are down.

a) (5 pts) Give the matrix of transition probabilities for this continuous time Markov chain.

b) (5 pts) Give the parameters of the exponential times the process stays in each state (v_0, v_1,v_2 and v_3).

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Questions 11-40 below are worth 1 point each. Mark each one T (for True), or F (for False).

11. ______ For state i of a Markov chain, let f_i denote the probability that starting in state i, the process will return to i. State i is said to be recurrent if $f_i=1$ and transient if $f_i<1$.

12. ______ A Markov chain having period 1 is said to be aperiodic.

13. ______ If $P^n_{ij}>0$ for some $n\geq 0$, states i and j of a Markov chain are said to communicate.

14. ______ Suppose that state i of a Markov chain is transient. Then starting in state i, the expected number of returns to i is $1/(1-f_i)$.

15. ______ State i of a Markov chain is recurrent if and only if

$$\sum_{n=1}^{\infty} P^n_{ii} < \infty.$$

16. ______ A Markov Chain with just a single recurrent class is said to be inscrutable.

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17. _____ If state i of a Markov chain is recurrent and $i \leftrightarrow j$, then j is recurrent.

18. _____ Not all states in a finite state Markov Chain can be recurrent.

19. _____ For a Markov chain with state space the set of all positive integers, let $P_{i,i+1}=p$, $P_{i,i-1}=1-p$ for $i = 0, \pm1, \pm2, \ldots$. All states are transient.

20. _____ For an irreducible aperiodic Markov chain with a finite state space, a limiting transition probability matrix always exists.

21. _____ For a random variable with distribution function F and density f, define the failure rate as $r(t) = \dfrac{f(t)}{1-F(t)}$. For an exponential random variable, the failure rate is constant.

22. _____ Let $\{N(t): t\geq 0\}$ be a counting process. If $s<t$, then $N(s)\leq N(t)$.

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23. ______ Let $\{N(t): t\geq 0\}$ be a counting process. The process is said to possess stationary increments if the distribution of the number of events in an interval depends only on the length of the interval.

24. ______ Let $\{N(t): t\geq 0\}$ be a counting process. The process is said to possess independent increments if and only if the numbers of events in any two intervals are independent random variables.

25. ______ Let $\{N(t): t\geq 0\}$ be a homogeneous Poisson process with rate λ. Then the number of events in an interval of length s has a Poisson distribution with mean λs.

26. ______ The function $f(\cdot)$ is said to be $o(h)$ if and only if for all $\epsilon > 0$, $\lim_{n\to\infty} P\{|f(h) - h| > \epsilon\} = 0$.

27. ______ Let $\{N(t): t\geq 0\}$ be a homogeneous Poisson process with rate λ. Then $P\{N(h)=1\} = o(h)$.

28. ______ Let $\{N(t): t\geq 0\}$ be a homogeneous Poisson process with rate λ. The arrival times of the events are independent Poisson random variables.

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29. _____ Let $\{N(t): t \geq 0\}$ be a homogeneous Poisson process with rate λ. The waiting times for the events are independent Gamma random variables.

30. _____ Let $\{N(t): t \geq 0\}$ be a homogeneous Poisson process with rate λ. Given that n events occur in $(0,t]$, the arrival times are independent random variables, uniformly distributed over $(0,t]$.

31. _____ Let $\{N(t): t \geq 0\}$ be a homogeneous Poisson process with rate λ. Independently of everything else, events are classified as type 1 with probability p and type 2 with probability 1-p. Then $p = \int_0^t e^{-\lambda s} ds$

32. _____ Let $\{N_1(t): t \geq 0\}$ and $\{N_2(t): t \geq 0\}$ be independent homogeneous Poisson processes with respective rates λ_1 and λ_2. The probability that n events from process 1 occur before m events from process 2 is $\binom{n+m-1}{n}\left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right)^n\left(\frac{\lambda_2}{\lambda_1+\lambda_2}\right)^m$.

33. _____ Let $\{N(t): t \geq 0\}$ be a homogeneous Poisson process with rate λ. The inter-arrival times are independent and identically distributed exponential random variables with mean $1/\lambda$.

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34. ______ Let $\{N(t): t \geq 0\}$ be a homogeneous Poisson process with rate λ. Independently of everything else, events are classified as type 1 with probability p and type 2 with probability 1-p, except that the probability p depends on the time that the event arrived; that is, p=p(t). Then the number of events in (0,t] has a Poisson distribution with mean $\lambda \int_0^t p(s)ds$.

35. ______ Let $\{N(t): t \geq 0\}$ be a non-homogeneous Poisson process with intensity function $\lambda(t)$. This process does not have stationary increments.

36. ______ Let $\{X(t): t \geq 0\}$ be a compound Poisson process based upon a homogeneous Poisson process with rate λ, and a sequence of continuous random variables $\{Y_1, Y_2, \ldots\}$. Then for t>0, the random variable X(t) is neither discrete nor continuous.

37. ______ Let $\{N(t): t \geq 0\}$ be a non-homogeneous Poisson process with intensity function $\lambda(t)$. $P\{N(t+h)-N(t))=1\}$ is $\lambda(t)h + o(h)$

38. ______ Every integer-valued continuous time Markov chain can be expressed as a counting process.

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39. _____ For a continuous time Markov chain, the time that the process stays in each state is an exponential random variable.

40. _____ The homogeneous Poisson process is a pure death process.

TOTAL MARKS = 100 POINTS