Sample Questions: Censoring and Likelihood

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1. The Pareto density is $f(t|\theta) = \begin{cases} \frac{\theta}{t^{\theta+1}} & \text{for } t \geq 1 \\ 0 & \text{for } t < 1 \end{cases}$. $F(t|\theta) = 1 - \frac{1}{t^{\theta}}$

Let $(T_1, \delta_1), \ldots, (T_n, \delta_n)$ be a random sample from a Pareto distribution with right censoring. Find the maximum likelihood estimate of θ.

$$L(\theta) = \prod_{i=1}^{n} \left(\frac{\theta}{t_i^{\theta+1}} \right)^{\delta_i} \left(\frac{1}{t_i^{\theta}} \right)^{1-\delta_i}$$

$$\ell(\theta) = \sum_{i=1}^{n} \left(\delta_i \log \left(\frac{\theta}{t_i^{\theta+1}} \right) + (1-\delta_i) \log t_i^{-\theta} \right)$$

$$= \sum_{i=1}^{n} \left(\delta_i (\log\theta - (\theta+1)\log t_i) + (1-\delta_i)(-\theta)\log t_i \right)$$

$$= \left(\sum_{i=1}^{n} \delta_i \right) \log\theta - (\theta+1) \sum_{i=1}^{n} \delta_i \log t_i - \theta \sum_{i=1}^{n} (1-\delta_i) \log t_i$$

$$\ell'(\theta) = \frac{\sum_{i=1}^{n} \delta_i}{\theta} - \sum_{i=1}^{n} \delta_i \log t_i - \sum_{i=1}^{n} (1-\delta_i) \log t_i$$

$$= \frac{\sum \delta_i}{\theta} - \sum_{i=1}^{n} \delta_i \log t_i - \sum_{i=1}^{n} \log t_i + \sum_{i=1}^{n} \delta_i \log t_i$$

$$= \frac{\sum_{i=1}^{n} \delta_i}{\theta} - \sum_{i=1}^{n} \log t_i \stackrel{\text{set}}{=} 0 \Rightarrow \frac{\sum_{i=1}^{n} \delta_i}{\theta} = \sum_{i=1}^{n} \log t_i$$

$$\Rightarrow \theta = \frac{\sum_{i=1}^{n} \delta_i}{\sum_{i=1}^{n} \log t_i}$$

2. Find the standard error of $\widehat{\theta}$.

$$\ell'(\theta) = \theta^{-1} \sum_{i=1}^{n} \delta_i - \sum_{i=1}^{n} \log t_i$$

$$\ell''(\theta) = (-1)\theta^{-2} \sum_{i=1}^{n} \delta_i = -\frac{\sum \delta_i}{\theta^2}$$

$$\hat{v}_n = \frac{-1}{\ell''(\hat{\theta})} = \frac{\hat{\theta}_n^2}{\sum_{i=1}^{n} \delta_i}$$

$$se = \frac{\hat{\theta}_n}{\sqrt{\sum_{i=1}^{n} \delta_i}}$$

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http://www.utstat.toronto.edu/~brunner/oldclass/312s19