STA 312 Assignment 4

Q1-3: See sample problems

Q4) Weibull

a) $S(t) = \int_t^\infty \alpha\lambda(\lambda x)^{\alpha-1} e^{-(\lambda x)^\alpha}\, dx$

$u = (\lambda x)^\alpha \quad du = \alpha(\lambda x)^{\alpha-1}\lambda\, dx$

$$\begin{array}{c|c} x & \\ \infty & \infty \\ t & (\lambda t)^\alpha \end{array}$$

$= \int_{(\lambda t)^\alpha}^\infty e^{-u}\, du = e^{-(\lambda t)^\alpha}$

b) $h(t) = \frac{f(t)}{S(t)} = \frac{\alpha\lambda(\lambda t)^{\alpha-1}\cancel{e^{-(\lambda t)^\alpha}}}{\cancel{e^{-(\lambda t)^\alpha}}}$

$= \alpha\lambda^\alpha t^{\alpha-1}$

c) $h(t)$ is increasing iff $\log h(t)$ is increasing

$\log h(t) = \log(\alpha\lambda^\alpha) + (\alpha-1)\log t$ &

$\frac{d}{dt}\log h(t) = 0 + \frac{(\alpha-1)}{t}$, so

$h(t)\downarrow$ for $\alpha<1$, $h(t)$ constant at $\alpha=1$ &
$h(t)\uparrow$ for $\alpha>1$

d) As $t\to\infty$, $h(t)\downarrow 0$ for $\alpha<1$
$h(t)\to\lambda$ for $\alpha=1$
$h(t)\uparrow\infty$ for $\alpha>1$

Q5 a) $S(t) = e^{-\int_0^t h(x)\,dx}$

$$= \mathrm{Exp}\left\{-\int_0^t (x-2)^2\,dx\right\} \quad \begin{array}{l} u = x-2 \\ du = dx \end{array} \quad \begin{array}{c|c} x & u \\ \hline t & t-2 \\ 0 & -2 \end{array}$$

$$= \mathrm{Exp}\left\{-\int_{-2}^{t-2} u^2\,du\right\} = e^{-\frac{u^3}{3}\Big|_{-2}^{t-2}}$$

$$= \mathrm{Exp}\left\{-\tfrac{1}{3}\left((t-2)^3 - -8\right)\right\} = \mathrm{Exp}\left\{-\tfrac{1}{3}\left((t-2)^3+8\right)\right\}$$

b) For $t > 0$, $f(t) = \frac{d}{dt}\left(1 - S(t)\right)$

$$= (-1)\,\mathrm{Exp}\left\{-\tfrac{1}{3}\left((t-2)^3+8\right)\right\}\left(-\tfrac{1}{\cancel{3}}\cancel{3}(t-2)^2\right)$$

$$\text{So } f(t) = \begin{cases} e^{-8/3}(t-2)^2 e^{-\frac{1}{3}(t-2)^3} & \text{for } t \geq 0 \\ 0 & \text{for } t < 0 \end{cases}$$

Q6 $X \sim \mathrm{EXP}(1)$, $Y = \log X$.

a) $\mathbb{R}$

b) $f_Y(y) = \frac{d}{dy} P(Y \le y) = \frac{d}{dy} P(\log X \le y) = \frac{d}{dy} P(X \le e^y)$

$= \frac{d}{dy} F_X(e^y) = f_X(e^y) \cdot e^y$

$= e^{-e^y} e^y = e^{y - e^y}$ for $-\infty < y < \infty$

c) Plot with R

d) $F_Y(y) = P(Y \le y) = P(\log X \le y) = F_X(e^y)$

$= 1 - e^{-e^y} \stackrel{set}{=} \frac{1}{2}$

$\Longleftrightarrow e^{-e^y} = \frac{1}{2} \Longleftrightarrow e^{e^y} = 2 \Rightarrow y = \log(\log 2)$

$= -0.3665$

e) $\log f(y) = y - e^y$, & $\frac{d}{dy} \log f(y) = 1 - e^y$

$1 - e^y = 0 \Longleftrightarrow e^y = 1 \Longleftrightarrow y = \log(1) = 0$

2nd derivative is $-e^y < 0$ CCD $\cap$ MAX

So the mode is zero.

(f) Using (d), $S(t) = 1 - F(t) = e^{-e^y}$

Q6g (i) $M(t) = E(e^{Yt})$

$$= \int_{-\infty}^{\infty} e^{yt} e^{y - e^y} dy = \int_{-\infty}^{\infty} e^{yt} e^{y} e^{-e^y} dy$$

$u = e^y$
$du = e^y dy$

y	$u = e^y$
∞	∞
$-\infty$	0

$$= \int_0^{\infty} u^t e^{-u} du = \int_0^{\infty} e^{-u} u^{(t+1)-1} du$$

$= \Gamma(t+1)$ No problem in an interval around $t=0$

(ii) $\frac{d}{dt} M(t) \Big|_{t=0} = \Gamma'(1)$ has no closed form, but

$$\text{digamma}(x) = \frac{d}{dx} \log \Gamma(x) = \frac{\Gamma'(x)}{\Gamma(x)}, \text{ so}$$

$$\text{digamma}(1) = \frac{\Gamma'(1)}{\Gamma(1)} = \frac{\Gamma'(1)}{1} = \Gamma'(1)$$

From R

Q7 $f_X(x) = \frac{d}{dx} F_X(x) = \frac{d}{dx} P(X \le x) = \frac{d}{dx} P(\sigma Z + \mu \le x)$

$= \frac{d}{dx} P(\sigma Z \le x - \mu) = \frac{d}{dx} P(Z \le \frac{x-\mu}{\sigma})$

$= \frac{d}{dx} F_Z(\frac{x-\mu}{\sigma}) = f_Z(\frac{x-\mu}{\sigma}) \cdot \frac{1}{\sigma}$

$= \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}(\frac{x-\mu}{\sigma})^2}$

$N(\mu, \sigma^2)$

Q8 Proceed exactly as above except for the last line.

Q9 $f(y|\mu,\sigma) = \frac{1}{\sigma} \operatorname{Exp}\left\{ \left(\frac{y-\mu}{\sigma}\right) - e^{\left(\frac{y-\mu}{\sigma}\right)} \right\}$

No, $E(Y) = E(\sigma Z + \mu) = \sigma E(Z) + \mu$

$= \mu - \sigma\gamma$, where $\gamma =$
is Euler's constant

Q10

(a) $f_Y(y) = \frac{d}{dy} P(Y \le y) = \frac{d}{dy} P(\log T \le y)$

$= \frac{d}{dy} P(T \le e^y) = \frac{d}{dy} F_T(e^y) = f_T(e^y) \cdot e^y$

$= \alpha \lambda (\lambda e^y)^{\alpha-1} \operatorname{Exp}\{-(\lambda e^y)^\alpha\} e^y$

$= \alpha \lambda^\alpha e^{y(\alpha-1)} \operatorname{Exp}\{-\lambda^\alpha e^{\alpha y}\} e^y$

$= \alpha \lambda^\alpha e^{\alpha y} \cancel{e^{-y}} \operatorname{Exp}\{-\lambda^\alpha e^{\alpha y}\} \cancel{e^y}$

(b) with $\sigma = \frac{1}{\alpha} \iff \alpha = \frac{1}{\sigma}$ and $\mu = -\log \lambda$

$\iff \lambda = e^{-\mu}$,

$f_Y(y \mid \mu, \sigma) = \frac{1}{\sigma} e^{-\frac{\mu}{\sigma}} e^{y/\sigma} \operatorname{Exp}\{-e^{-\mu/\sigma} e^{y/\sigma}\}$

$= \frac{1}{\sigma} e^{\left(\frac{y-\mu}{\sigma}\right)} \operatorname{Exp}\{-e^{\left(\frac{y-\mu}{\sigma}\right)}\}$

$= \frac{1}{\sigma} \operatorname{Exp}\{\left(\frac{y-\mu}{\sigma}\right) - e^{\left(\frac{y-\mu}{\sigma}\right)}\}$

Same as Q9.

Q11

(a) $S(t) = \int_t^{\infty} \frac{1}{\sigma} \operatorname{Exp}\left\{ \left(\frac{y-\mu}{\sigma}\right) - e^{\left(\frac{y-\mu}{\sigma}\right)} \right\} dy$

$$\left(\begin{array}{ll} z = \frac{y-\mu}{\sigma} & \begin{array}{c|c} y & z \\ \hline \infty & \infty \\ t & \left(\frac{t-\mu}{\sigma}\right) \end{array} \\ dz = \frac{1}{\sigma}\, dy & \end{array} \right)$$

$$= \int_{\left(\frac{t-\mu}{\sigma}\right)}^{\infty} e^{z} e^{-e^{z}} dz$$

$$\left(\begin{array}{ll} x = e^{z} & \begin{array}{c|c} z & x \\ \hline \infty & \infty \\ \left(\frac{t-\mu}{\sigma}\right) & e^{\left(\frac{t-\mu}{\sigma}\right)} \end{array} \\ dx = e^{z}\, dz & \end{array} \right)$$

$$= \int_{e^{\left(\frac{t-\mu}{\sigma}\right)}}^{\infty} e^{-x} dx = e^{-e^{\left(\frac{t-\mu}{\sigma}\right)}}$$

(b) $h(t) = \dfrac{f(t)}{S(t)} = \dfrac{\frac{1}{\sigma} e^{\left(\frac{t-\mu}{\sigma}\right)} \cancel{e^{-e^{\left(\frac{t-\mu}{\sigma}\right)}}}}{\cancel{e^{-e^{\left(\frac{t-\mu}{\sigma}\right)}}}}$

$$= \frac{1}{\sigma} e^{\left(\frac{t-\mu}{\sigma}\right)}$$

Q11 c $\frac{d}{dy}\log f(y) = \frac{d}{dy}\left(\log \sigma^{-1} + \left(\frac{y-\mu}{\sigma}\right) - e^{\left(\frac{y-\mu}{\sigma}\right)}\right)$

$$= \frac{1}{\sigma} - e^{\left(\frac{y-\mu}{\sigma}\right)} \cdot \frac{1}{\sigma} \overset{\text{set}}{=} 0$$

$$\iff \frac{1}{\sigma} = \frac{1}{\sigma} e^{\left(\frac{y-\mu}{\sigma}\right)} \iff e^{\left(\frac{y-\mu}{\sigma}\right)} = 1$$

$$\iff \log e^{\left(\frac{y-\mu}{\sigma}\right)} = \log(1) = 0$$

$$\iff \frac{y-\mu}{\sigma} = 0 \iff y = \mu$$

$$\frac{d^2 \log f(y)}{\partial y^2} = -\frac{1}{\sigma^2} e^{\left(\frac{y-\mu}{\sigma}\right)} < 0 \quad \text{CCD} \frown \text{MAX}$$

So the mode is μ

(d) $S(m) = \frac{1}{2}$; that is, $e^{-e^{\left(\frac{m-\mu}{\sigma}\right)}} = \frac{1}{2}$

$$\iff e^{e^{\left(\frac{m-\mu}{\sigma}\right)}} = 2 \iff \frac{m-\mu}{\sigma} = \log(\log 2)$$

$$\iff m - \mu = \sigma \log(\log 2)$$

$$\iff m = \sigma \log(\log 2) + \mu$$ Median

11e No need to go through the moment-generating function again

$$E(Y) = E(\sigma Z + \mu) = \sigma E(Z) + \mu$$
$$= \sigma(-\gamma) + \mu = \mu - \sigma\gamma$$

where γ is Euler's constant

(f) $Var(Y) = Var(\sigma Z + \mu) = \sigma^2 Var(Z)$
$$= \sigma^2 \left(\frac{\pi^2}{6}\right)$$