Contingency Tables: Part Two*

STA 312: Fall 2022

Suggested Reading: Chapter 2

- Read Section 2.6 about Fisher's exact test
- Read Section 2.7 about multi-dimensional tables and Simpson's paradox.

Contents

1 Testing for the Product Multinomial

Testing Association for the Product Multinomial

Prospective and retrospective designs

Prospective design:

- A conditional multinomial in each row
- I independent random samples, one for each value of X
- Likelihood is a product of I multinomials
- Null hypothesis is that all I sets of conditional probabilities are the same.

A retrospective design is just like this, but with rows and columns reversed.

*See last slide for copyright information.

Null hypothesis is no differences among the I vectors of conditional probabilities

	Attack	Stroke	Both	Neither	Total
Drug					n_{1+}
Drug and Exercise					n_{2+}
Total	n_{+1}	n_{+2}	n_{+3}	n_{+4}	n

- Both n_{1+} and n_{2+} are fixed by the design. They are *sample sizes.*
- Under H_0, MLE of the (common) conditional probability is the marginal sample proportion:
$$\widehat{\pi}_{ij} = p_{+j} = \frac{n_{+j}}{n}$$
- And the expected cell frequency is just
$$\widehat{\mu}_{ij} = n_{i+}\,\widehat{\pi}_{ij} = n_{i+}\,\frac{n_{+j}}{n} = \frac{n_{i+}n_{+j}}{n}.$$

Expected frequencies are the same!

For testing both independence and testing equal conditional probabilities,
$$\widehat{\mu}_{ij} = \frac{n_{i+}n_{+j}}{n}.$$
The degrees of freedom are the same too. For the product multinomial,

- There are $I(J-1)$ free parameters in the unconstrained model.
- There are $J-1$ free parameters under the null hypothesis.
- H_0 imposes $I(J-1)-(J-1)=(I-1)(J-1)$ constraints on the parameter vector.
- So $df = (I-1)(J-1)$.

	Attack	Stroke	Both	Neither	Total
Drug					n_{1+}
Drug and Exercise					n_{2+}
Total	n_{+1}	n_{+2}	n_{+3}	n_{+4}	n

This is very fortunate

- The cross-sectional, prospective and retrospectives are different from one another conceptually.
- The multinomial and product-multinomial models are different from one another technically.
- But the tests for relationship between explanatory and response variables are 100% the same.
- Same expected frequencies and same degrees of freedom.
- Therefore we get the same test statistics and p-values.

2 Fisher's Exact Test

Fisher's Exact Test

- Everything so far is based on large-sample theory.
- What if the sample is small?
- Fisher's exact test is good for 2×2 tables.
- There are extensions for larger tables.

Fisher's exact test is a permutation test

		Y = 1	Y = 2	
X	1	x	$a - x$	a
	2	$b - x$	$n - a - b + x$	$n - a$
		b	$n - b$	n

- Think of a data file with 2 columns, X and Y, filled with ones and twos.
- X has a ones and Y has b ones.
- Calculate the estimated odds ratio $\widehat{\theta}$.
- If X and Y are unrelated, all possible pairings of X and Y values should be equally likely.
- There are $n!$ ways to order the X values, and for each of these, $n!$ ways to order the Y values.

Idea of a permutation test

		Y = 1	Y = 2	
X	1	x	$a - x$	a
	2	$b - x$	$n - a - b + x$	$n - a$
		b	$n - b$	n

- There are $(n!)^2$ ways to arrange the X and Y values.
- For what fraction of these is the (estimated) odds ratio
  - Greater than or equal to $\widehat{\theta}$ (Upper tail p-value)
  - Less than or equal to $\widehat{\theta}$ (Lower tail p-value)

  For a 2-sided test, add the probabilities of all the tables less likely than or equally likely to the one we have observed. (This is what R does.)

Nice idea, but hard to compute. Fisher thought of it *and* simplified it.

Let us count together

		Y		
		1	2	
X	1	x	$a-x$	a
	2	$b-x$	$n-a-b+x$	$n-a$
		b	$n-b$	n

- The $n!$ permutations of 1s and 2s have lots of repeats that look the same.
- There are $\binom{n}{a}$ ways to choose which cases have $X = 1$.
- For each of these, there are $\binom{n}{b}$ ways to choose which cases have $Y = 1$.
- So the total number of 2×2 tables with n observations, $n_{1+} = a$ and $n_{+1} = b$ is $\binom{n}{a}\binom{n}{b}$.
- Of these, the number of ways to get the values in the table is just the multinomial coefficient

$$\binom{n}{x \quad a-x \quad b-x \quad n-a-b+x} = \frac{n!}{x!(a-x)!(b-x)!(n-a-b+x)!}.$$

Hypergeometric probability

		Y		
		1	2	
X	1	x	$a-x$	$a = n_{1+}$
	2	$b-x$	$n-a-b+x$	$n-a = n_{2+}$
		$b = n_{+1}$	$n-b = n_{+2}$	n

Dividing the number of ways to get $n_{11} = x$ by the total number of equally likely outcomes,

$$\begin{aligned}
P(n_{11} = x) &= \frac{\binom{n}{x \;\; a-x \;\; b-x \;\; n-a-b+x}}{\binom{n}{a}\binom{n}{b}} \\
&= \frac{\frac{n!}{x!(a-x)!(b-x)!(n-a-b+x)!}}{\frac{n!}{a!(n-a)!}\,\frac{n!}{b!(n-b)!}} \\
&= \frac{\binom{a}{x}\binom{n-a}{b-x}}{\binom{n}{b}} \\
&= \frac{\binom{n_{1+}}{n_{11}}\binom{n_{2+}}{n_{+1}-n_{11}}}{\binom{n}{n_{+1}}} \qquad \text{(Eq. 2.11, p. 46)}
\end{aligned}$$

Adding up the probabilities

Always remembering that a, b and n are fixed

		Y		
		1	2	
X	1	x	$a-x$	a
	2	$b-x$	$n-a-b+x$	$n-a$
		b	$n-b$	n

- Fortunately, $\theta(x)$ is an increasing function of x (differentiate).
- So, tables with larger x values than the one observed also have greater sample odds ratios. Add $P(n_{11} = x)$ over x to get tail probabilities.
- Range of x:
  - $x \leq \min(a, b)$
  - $n_{22} = n - a - b + x \geq 0$, so $x \geq a + b - n$.
  - Thus, x ranges from $\max(0, a + b - n)$ to $\min(a, b)$.

Example: Sinking of the the Titanic

```
> # help(Titanic)
> dimnames(Titanic)

$Class
[1] "1st"  "2nd"  "3rd"  "Crew"

$Sex
[1] "Male"   "Female"

$Age
[1] "Child" "Adult"

$Survived
[1] "No"  "Yes"

> # Women in 1st class vs Women in crew
>
> ladies = Titanic[c(1,4),2,2,]
```

Just the ladies

```
> ladies
      Survived
Class  No Yes
  1st   4 140
  Crew  3  20
> 140/144 # Rich ladies
[1] 0.9722222
> 20/23   # Cleaning ladies
[1] 0.8695652
> X2 = chisq.test(ladies,correct=F); X2
Warning message:
In chisq.test(ladies, correct = F) :
  Chi-squared approximation may be incorrect
```

```
Pearson's Chi-squared test

data:  ladies
X-squared = 5.2043, df = 1, p-value = 0.02253
```

Check the expected frequencies

```
> X2$expected
      Survived
Class         No       Yes
  1st  6.0359281 137.96407
  Crew 0.9640719  22.03593

>
> fisher.test(ladies)

Fisher's Exact Test for Count Data

data:  ladies
p-value = 0.05547
alternative hypothesis: true odds ratio is not equal to 1
95 percent confidence interval:
 0.03027561 1.41705937
sample estimates:
odds ratio
 0.1935113
```

Conclusion

Though a higher percentage of women in first class survived than female crew, it could have been due to chance.

Fisher's exact test makes sense even without the pretending we have a random sample

You could say

- Assume that status on the ship for these women (First Class passenger vs. crew) is fixed. It was what it was.
- Survival also was what it was.
- Given this, is the observed *pairing* of status and survival an unusual one?
- That is, for what fraction of the possible pairings is the status difference in survival as great or greater than the one we have observed?
- A little over 5%? That's a bit unusual, but perhaps not *very* unusual.
- **There is not even any need to talk about probability.**

3 Tables of Higher Dimension

Tables of Higher Dimension: Conditional independence

- Suppose X and Y are related.
- Are X and Y related *conditionally* on the value of W?
- One sub-table for each value of W.
- X and Y can easily be related unconditionally, but still be conditionally independent.
- Example: Among adults 18 and older, X =Tattoos and Y =Grey hair.
- Need a 3-way table, showing the relationship of tattoos and grey hair separately for each age group.
- Speak of the relationship between X and Y "controlling for" W, or "allowing for" W.

Was UC Berkeley discriminating against women?

Data from the 1970s

Data in a 3-dimensional array: Variables are

- Sex of the person applying for graduate study
- Department to which the person applied
- Whether or not the person was admitted

Berkeley data

```
> ##########################################################
> #   More than one Explanatory Variable at once           #
> #   data()  to list the nice data sets that come with R  #
> #   help(UCBAdmissions)                                  #
> ##########################################################
> dim(UCBAdmissions)
[1] 2 2 6
> dimnames(UCBAdmissions)
$Admit
[1] "Admitted" "Rejected"

$Gender
[1] "Male"   "Female"

$Dept
[1] "A" "B" "C" "D" "E" "F"

> # Look at gender by admit.
> # Apply sum to rows and columns, obtaining the marginal freqs.
> sexadmit = apply(UCBAdmissions,c(1,2),sum)
```

Sex by Admission

```
> sexadmit

          Gender
Admit      Male Female
  Admitted 1198    557
  Rejected 1493   1278
> sexadmit = t(sexadmit); sexadmit
        Admit
Gender   Admitted Rejected
  Male       1198     1493
  Female      557     1278
> rowmarg = apply(sexadmit,1,sum); rowmarg
  Male Female
  2691   1835
> percentadmit = 100 * sexadmit[,1]/rowmarg ; percentadmit
    Male   Female
44.51877 30.35422
```

It certainly looks suspicious.

Test sex by admission

```
> chisq.test(sexadmit,correct=F)

Pearson's Chi-squared test

data:  sexadmit
X-squared = 92.2053, df = 1, p-value < 2.2e-16

> fisher.test(sexadmit)   # Gives same p-value

Fisher's Exact Test for Count Data

data:  sexadmit
p-value < 2.2e-16
alternative hypothesis: true odds ratio is not equal to 1
95 percent confidence interval:
 1.621356 2.091246
sample estimates:
odds ratio
  1.840856
```

But look at the whole table

```
> UCBAdmissions
, , Dept = A

          Gender
```

```
Admit      Male Female
  Admitted  512     89
  Rejected  313     19

, , Dept = B

          Gender
Admit      Male Female
  Admitted  353     17
  Rejected  207      8
```

Berkeley table continued

```
, , Dept = C

          Gender
Admit      Male Female
  Admitted  120    202
  Rejected  205    391

, , Dept = D

          Gender
Admit      Male Female
  Admitted  138    131
  Rejected  279    244
```

Berkeley table continued some more

```
, , Dept = E

          Gender
Admit      Male Female
  Admitted   53     94
  Rejected  138    299

, , Dept = F

          Gender
Admit      Male Female
  Admitted   22     24
  Rejected  351    317
```

Look at Department A

```
> # Just Department A
> JustA = t(UCBAdmissions[,,1]); JustA
        Admit
```

```
Gender   Admitted Rejected
  Male        512      313
  Female       89       19
> JustA[1,1]/sum(JustA[1,]) # Men
[1] 0.6206061
> JustA[2,1]/sum(JustA[2,]) # Women
[1] 0.8240741
> chisq.test(UCBAdmissions[,,1],correct=F)

Pearson's Chi-squared test

data:  UCBAdmissions[, , 1]
X-squared = 17.248, df = 1, p-value = 3.28e-05
```

Women are more likely to be admitted.

Summarize analyses of sub-tables

Just the code, for reference

```
# Summarize analyses of sub-tables: Loop over departments
# Sum of chi-squared values in X2
ndepts = dim(UCBAdmissions)[3]
gradschool=NULL; X2=0
for(j in 1:ndepts)
    {
    dept = dimnames(UCBAdmissions)$Dept[j] # A B C etc.
    tabl = t(UCBAdmissions[,,j]) # All rows, all cols, level j
    Rowmarg = apply(tabl,1,sum)
    Percentadmit = round( 100*tabl[,1]/Rowmarg ,1)
    per = round(Percentadmit,2)
    Test = chisq.test(tabl,correct=F)
    tstat = round(Test$statistic,2); pval = round(Test$p.value,5)
    gradschool = rbind(gradschool,c(dept,Percentadmit,tstat,pval))
    X2 = X2+Test$statistic
    } # Next Department
colnames(gradschool) = c("Dept","%MaleAcc","%FemAcc","Chisq","p-value")
noquote(gradschool) # Print character strings without quote marks
```

Simpson's paradox

```
> noquote(gradschool) # Print character strings without quote marks

     Dept %MaleAcc %FemAcc Chisq p-value
[1,] A    62.1     82.4    17.25 3e-05
[2,] B    63       68      0.25  0.61447
[3,] C    36.9     34.1    0.75  0.38536
[4,] D    33.1     34.9    0.3   0.58515
[5,] E    27.7     23.9    1     0.31705
[6,] F    5.9      7       0.38  0.53542
```

Overall test of conditional independence

Add the chi-squared values and add the degrees of freedom.

```
> # Overall test of conditional independence
> names(X2) = "Pooled Chi-square"
> df = ndepts ; names(df)="df"
> pval=1-pchisq(X2,df)
> names(pval) = "P-value"
> print(c(X2,df,pval))
Pooled Chi-square                df           P-value
     19.938413378       6.000000000       0.002840164
```

Conclusion: Gender and admission are *not* conditionally independent. From the preceding slide, we see it comes from Department A's being more likely to admit women than men.

Track it down

Make a table showing Department, Number of applicants, Percent female applicants and Percent of applicants admitted.

```
> # What's happening?
> whoapplies = NULL
> for(j in 1:ndepts)
+     {
+     dept = dimnames(UCBAdmissions)$Dept[j]; names(dept) = "Dept"
+     tabl = t(UCBAdmissions[,,j]) # All rows, all cols, level j
+     nj = sum(tabl); names(nj)=" n "
+     mf = apply(tabl,1,sum); femapp = round(100*mf[2]/nj,2)
+     succ = apply(tabl,2,sum); getin = round(100*succ[1]/nj,2)
+     whoapplies = rbind(whoapplies,c(dept,nj,femapp,getin))
+     } # Next Department
>
```

Now it's in a table called `whoapplies`.

The explanation

```
> noquote(whoapplies)

     Dept  n  Female Admitted
[1,] A    933 11.58  64.42
[2,] B    585 4.27   63.25
[3,] C    918 64.6   35.08
[4,] D    792 47.35  33.96
[5,] E    584 67.29  25.17
[6,] F    714 47.76  6.44
```

Departments with a higher acceptance rate have a higher percentage of male applicants.

Does this mean that the University of California at Berkeley was *not* discriminating against women?

- By no means. Why does a department admit very few applicants relative to the number who apply?
- Because they do not have enough professors and other resources to offer more classes.
- This implies that the departments popular with men were getting more resources, relative to the level of interest measured by number of applicants.
- Why? Maybe because men were running the show.
- The "show," definitely includes the U. S. military, which funds a lot of engineering and similar stuff at big American universities.

Some uncomfortable truths

- Especially for non-experimental studies, statistical analyses involving just one explanatory variable at a time can be very misleading.
- When you include a new variable in an analysis, the results could get weaker, they could get stronger, or they could reverse direction — all depending upon the inter-relations of the explanatory variables and the response variable.
- Failing to include important explanatory variables in observational studies is a common source of bias.
- Ask: "Did you control for ..."

At least it's a start

- We have seen one way to "control" for potentially misleading variables (sometimes called "confounding variables").
- It's *control by sub-division*, in which you examine the relationship in question separately for each value of a control variable or variables.
- We have a good way of pooling the tests within each level of the control variable, to obtain a test of conditional independence.
- There's also model-based control, which is coming next.