Logistic Regression with more than two outcomes

Think of k-1 dummy variables for the dependent variable

Model for three categories

$$\log\left(\frac{\pi_1}{\pi_3}\right) = \beta_{0,1} + \beta_{1,1}x_1 + \ldots + \beta_{p-1,1}x_{p-1}$$

$$\log\left(\frac{\pi_2}{\pi_3}\right) = \beta_{0,2} + \beta_{1,2}x_1 + \ldots + \beta_{p-1,2}x_{p-1}$$

Need *k-1* **generalized logits** to represent a dependent variable with *k* categories

Meaning of the regression coefficients

$$\log\left(\frac{\pi_1}{\pi_3}\right) = L_1$$

$$\log\left(\frac{\pi_2}{\pi_3}\right) = L_2$$

A positive regression coefficient for logit *j* means that higher values of the independent variable are associated with greater chances of response category *j*, compared to the reference category.

Solve for the probabilities

$$\log\left(\frac{\pi_1}{\pi_3}\right) = L_1 \qquad \frac{\pi_1}{\pi_3} = e^{L_1}$$

so

$$\log\left(\frac{\pi_2}{\pi_3}\right) = L_2 \qquad \frac{\pi_2}{\pi_3} = e^{L_2}$$

$$\pi_1 = \pi_3 e^{L_1}$$

So

$$\pi_2 = \pi_3 e^{L_2}$$

Three linear equations in 3 unknowns

$$
\begin{aligned}
\pi_1 &= \pi_3 e^{L_1} \\
\pi_2 &= \pi_3 e^{L_2} \\
\pi_1 + \pi_2 + \pi_3 &= 1
\end{aligned}
$$

Solution

$$\pi_1 = \frac{e^{L_1}}{1 + e^{L_1} + e^{L_2}}$$

$$\pi_2 = \frac{e^{L_2}}{1 + e^{L_1} + e^{L_2}}$$

$$\pi_3 = \frac{1}{1 + e^{L_1} + e^{L_2}}$$

Example: One Binary Independent Variable

$$\log\left(\frac{\pi_1}{\pi_3}\right) = \beta_{0,1} + \beta_{1,1}x = L_1$$

$$\log\left(\frac{\pi_2}{\pi_3}\right) = \beta_{0,2} + \beta_{1,2}x = L_2$$

	$Y = 1$	$Y = 2$	$Y = 3$
$x = 0$	$\frac{e^{\beta_{01}}}{1+e^{\beta_{01}}+e^{\beta_{02}}}$	$\frac{e^{\beta_{02}}}{1+e^{\beta_{01}}+e^{\beta_{02}}}$	$\frac{1}{1+e^{\beta_{01}}+e^{\beta_{02}}}$
$x = 1$	$\frac{e^{\beta_{01}+\beta_{11}}}{1+e^{\beta_{01}+\beta_{11}}+e^{\beta_{02}+\beta_{12}}}$	$\frac{e^{\beta_{02}+\beta_{12}}}{1+e^{\beta_{01}+\beta_{11}}+e^{\beta_{02}+\beta_{12}}}$	$\frac{1}{1+e^{\beta_{01}+\beta_{11}}+e^{\beta_{02}+\beta_{12}}}$

In general, solve *k* equations in *k* unknowns

$$
\begin{aligned}
\pi_1 &= \pi_k e^{L_1} \\
&\vdots \\
\pi_{k-1} &= \pi_k e^{L_{k-1}} \\
\pi_1 + \cdots + \pi_k &= 1
\end{aligned}
$$

General Solution

$$
\begin{aligned}
\pi_1 &= \frac{e^{L_1}}{1+\sum_{j=1}^{k-1} e^{L_j}} \\
\pi_2 &= \frac{e^{L_2}}{1+\sum_{j=1}^{k-1} e^{L_j}} \\
&\vdots \\
\pi_{k-1} &= \frac{e^{L_{k-1}}}{1+\sum_{j=1}^{k-1} e^{L_j}} \\
\pi_k &= \frac{1}{1+\sum_{j=1}^{k-1} e^{L_j}}
\end{aligned}
$$

Using the solution, one can

- Calculate the probability of obtaining the observed data as a function of the regression coefficients: Get maximum likelihood estimates (*β-hat* values)
- From maximum likelihood estimates, get tests and confidence intervals
- Using *β-hat* values in L_j, estimate probabilities of category membership for any set of x values.