Logistic Regression with more than two outcomes

Think of k-1 dummy variables for the dependent variable

Model for three categories $\log\left(\frac{\pi_1}{\pi_3}\right) = \beta_{0,1} + \beta_{1,1}x_1 + \ldots + \beta_{p-1,1}x_{p-1}$ $\log\left(\frac{\pi_2}{\pi_3}\right) = \beta_{0,2} + \beta_{1,2}x_1 + \ldots + \beta_{p-1,2}x_{p-1}$

Need *k-1* generalized logits to represent a dependent variable with *k* categories

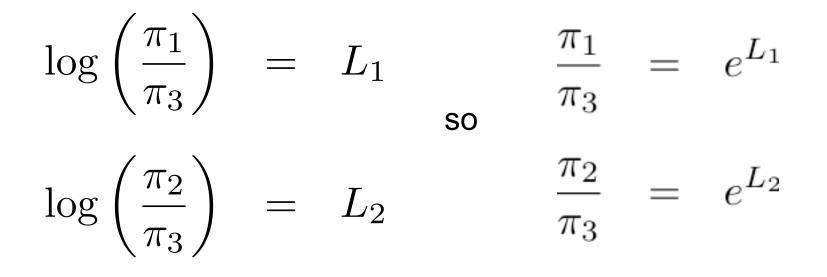
Meaning of the regression coefficients

$$\log\left(\frac{\pi_1}{\pi_3}\right) = L_1$$

$$\log\left(\frac{\pi_2}{\pi_3}\right) = L_2$$

A positive regression coefficient for logit *j* means that higher values of the independent variable are associated with greater chances of response category *j*, compared to the reference category.

Solve for the probabilities



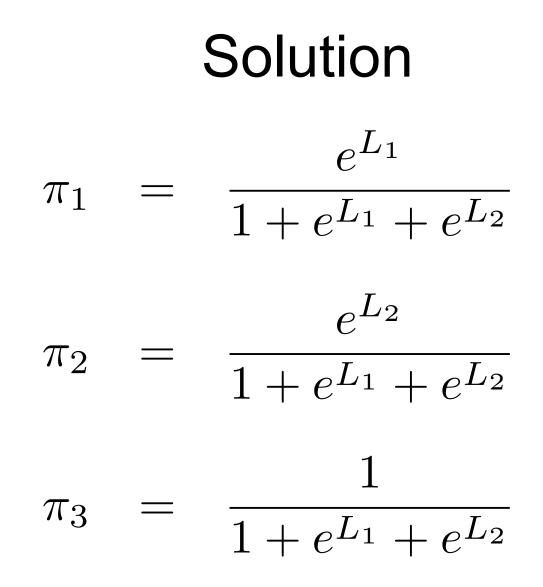
$$\pi_1 = \pi_3 e^{L_1}$$
 So
$$\pi_2 = \pi_3 e^{L_2}$$

Three linear equations in 3 unknowns

$$\pi_1 = \pi_3 e^{L_1}$$

$$\pi_2 = \pi_3 e^{L_2}$$

 $\pi_1 + \pi_2 + \pi_3 = 1$



Example: One Binary Independent Variable

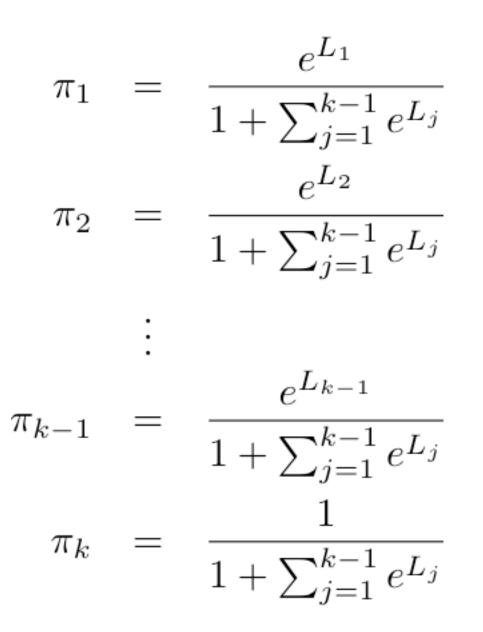
$$\log\left(\frac{\pi_1}{\pi_3}\right) = \beta_{0,1} + \beta_{1,1}x = L_1$$
$$\log\left(\frac{\pi_2}{\pi_3}\right) = \beta_{0,2} + \beta_{1,2}x = L_2$$

	Y = 1	Y = 2	Y = 3
x = 0	$\frac{e^{\beta_{01}}}{1 + e^{\beta_{01}} + e^{\beta_{02}}}$	$\frac{e^{\beta_{02}}}{1 + e^{\beta_{01}} + e^{\beta_{02}}}$	$\frac{1}{1+e^{\beta_{01}}+e^{\beta_{02}}}$
x = 1	$\frac{e^{\beta_{01}+\beta_{11}}}{1+e^{\beta_{01}+\beta_{11}}+e^{\beta_{02}+\beta_{12}}}$	$\frac{e^{\beta_{02}+\beta_{12}}}{1+e^{\beta_{01}+\beta_{11}}+e^{\beta_{02}+\beta_{12}}}$	$\frac{1}{1 + e^{\beta_{01} + \beta_{11}} + e^{\beta_{02} + \beta_{12}}}$

In general, solve k equations in k unknowns

 $\pi_1 = \pi_k e^{L_1}$ \vdots $\pi_{k-1} = \pi_k e^{L_{k-1}}$ $\pi_1 + \dots + \pi_k = 1$

General Solution



Using the solution, one can

- Calculate the probability of obtaining the observed data as a function of the regression coefficients: Get maximum likelihood estimates (β-hat values)
- From maximum likelihood estimates, get tests and confidence intervals
- Using β-hat values in L_j, estimate probabilities of category membership for any set of x values.