Random Vectors[1]

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Random Vectors and Matrices

See Chapter 3 of *Linear models in statistics* for more detail.

- A *random matrix* is just a matrix of random variables.
- Their joint probability distribution is the distribution of the random matrix.
- Random matrices with just one column (say, $p \times 1$) may be called *random vectors*.

Expected Value

The expected value of a random matrix is defined as the matrix of expected values. Denoting the $p \times c$ random matrix $\mathbf{X}$ by $[x_{i,j}]$,

$$E(\mathbf{X}) = [E(x_{i,j})].$$

Immediately we have natural properties like

If the random matrices $\mathbf{X}$ and $\mathbf{Y}$ are the same size,

$$\begin{aligned} E(\mathbf{X}+\mathbf{Y}) &= E([x_{i,j}+y_{i,j}]) \\ &= [E(x_{i,j}+y_{i,j})] \\ &= [E(x_{i,j})+E(y_{i,j})] \\ &= [E(x_{i,j})]+[E(y_{i,j})] \\ &= E(\mathbf{X})+E(\mathbf{Y}). \end{aligned}$$

Moving a constant matrix through the expected value sign

Let $\mathbf{A} = [a_{i,j}]$ be an $r \times p$ matrix of constants, while $\mathbf{X}$ is still a $p \times c$ random matrix. Then

$$
\begin{aligned}
E(\mathbf{AX}) &= E\left(\left[\sum_{k=1}^{p} a_{i,k} x_{k,j}\right]\right) \\
&= \left[E\left(\sum_{k=1}^{p} a_{i,k} x_{k,j}\right)\right] \\
&= \left[\sum_{k=1}^{p} a_{i,k} E(x_{k,j})\right] \\
&= \mathbf{A}E(\mathbf{X}).
\end{aligned}
$$

Similar calculations yield $E(\mathbf{AXB}) = \mathbf{A}E(\mathbf{X})\mathbf{B}$.

Variance-Covariance Matrices

Let $\mathbf{x}$ be a $p \times 1$ random vector with $E(\mathbf{x}) = \boldsymbol{\mu}$. The *variance-covariance matrix* of $\mathbf{x}$ (sometimes just called the *covariance matrix*), denoted by $cov(\mathbf{x})$, is defined as

$$cov(\mathbf{x}) = E\left\{(\mathbf{x} - \boldsymbol{\mu})(\mathbf{x} - \boldsymbol{\mu})'\right\}.$$

$cov(\mathbf{x}) = E\{(\mathbf{x} - \boldsymbol{\mu})(\mathbf{x} - \boldsymbol{\mu})'\}$

$$
\begin{aligned}
cov(\mathbf{x}) &= E\left\{ \begin{pmatrix} x_1 - \mu_1 \\ x_2 - \mu_2 \\ x_3 - \mu_3 \end{pmatrix} \begin{pmatrix} x_1 - \mu_1 & x_2 - \mu_2 & x_3 - \mu_3 \end{pmatrix} \right\} \\
&= E\left\{ \begin{pmatrix} (x_1 - \mu_1)^2 & (x_1 - \mu_1)(x_2 - \mu_2) & (x_1 - \mu_1)(x_3 - \mu_3) \\ (x_2 - \mu_2)(x_1 - \mu_1) & (x_2 - \mu_2)^2 & (x_2 - \mu_2)(x_3 - \mu_3) \\ (x_3 - \mu_3)(x_1 - \mu_1) & (x_3 - \mu_3)(x_2 - \mu_2) & (x_3 - \mu_3)^2 \end{pmatrix} \right\} \\
&= \begin{pmatrix} E\{(x_1 - \mu_1)^2\} & E\{(x_1 - \mu_1)(x_2 - \mu_2)\} & E\{(x_1 - \mu_1)(x_3 - \mu_3)\} \\ E\{(x_2 - \mu_2)(x_1 - \mu_1)\} & E\{(x_2 - \mu_2)^2\} & E\{(x_2 - \mu_2)(x_3 - \mu_3)\} \\ E\{(x_3 - \mu_3)(x_1 - \mu_1)\} & E\{(x_3 - \mu_3)(x_2 - \mu_2)\} & E\{(x_3 - \mu_3)^2\} \end{pmatrix} \\
&= \begin{pmatrix} Var(x_1) & Cov(x_1, x_2) & Cov(x_1, x_3) \\ Cov(x_1, x_2) & Var(x_2) & Cov(x_2, x_3) \\ Cov(x_1, x_3) & Cov(x_2, x_3) & Var(x_3) \end{pmatrix}.
\end{aligned}
$$

So, the covariance matrix $cov(\mathbf{x})$ is a $p \times p$ symmetric matrix with variances on the main diagonal and covariances on the off-diagonals.

Analogous to $Var(a\,x) = a^2\,Var(x)$

Let $\mathbf{x}$ be a $p \times 1$ random vector with $E(\mathbf{x}) = \boldsymbol{\mu}$ and $cov(\mathbf{x}) = \boldsymbol{\Sigma}$, while $\mathbf{A} = [a_{i,j}]$ is an $r \times p$ matrix of constants. Then

$$
\begin{aligned}
cov(\mathbf{Ax}) &= E\left\{(\mathbf{Ax} - \mathbf{A}\boldsymbol{\mu})(\mathbf{Ax} - \mathbf{A}\boldsymbol{\mu})'\right\} \\
&= E\left\{\mathbf{A}(\mathbf{x} - \boldsymbol{\mu})\,(\mathbf{A}(\mathbf{x} - \boldsymbol{\mu}))'\right\} \\
&= E\left\{\mathbf{A}(\mathbf{x} - \boldsymbol{\mu})(\mathbf{x} - \boldsymbol{\mu})'\mathbf{A}'\right\} \\
&= \mathbf{A}E\{(\mathbf{x} - \boldsymbol{\mu})(\mathbf{x} - \boldsymbol{\mu})'\}\mathbf{A}' \\
&= \mathbf{A}cov(\mathbf{x})\mathbf{A}' \\
&= \mathbf{A}\boldsymbol{\Sigma}\mathbf{A}'
\end{aligned}
$$

Positive definite is a natural assumption

For covariance matrices

- $cov(\mathbf{x}) = \boldsymbol{\Sigma}$
- $\boldsymbol{\Sigma}$ positive definite means $\mathbf{a}'\boldsymbol{\Sigma}\mathbf{a} > 0.$ for all $\mathbf{a} \neq \mathbf{0}$.
- $y = \mathbf{a}'\mathbf{x} = a_1 x_1 + \cdots + a_p x_p$ is a scalar random variable.
- $Var(y) = \mathbf{a}' cov(\mathbf{x}) \mathbf{a} = \mathbf{a}'\boldsymbol{\Sigma}\mathbf{a}$
- $\boldsymbol{\Sigma}$ positive definite just says that the variance of any (non-trivial) linear combination is positive.
- This is often what you want (but not always).

Matrix of covariances between two random vectors

Let $\mathbf{x}$ be a $p \times 1$ random vector with $E(\mathbf{x}) = \boldsymbol{\mu}_x$ and let $\mathbf{y}$ be a $q \times 1$ random vector with $E(\mathbf{y}) = \boldsymbol{\mu}_y$.

The $p \times q$ matrix of covariances between the elements of $\mathbf{x}$ and the elements of $\mathbf{y}$ is

$$cov(\mathbf{x}, \mathbf{y}) = E\left\{(\mathbf{x} - \boldsymbol{\mu}_x)(\mathbf{y} - \boldsymbol{\mu}_y)'\right\}.$$

Adding a constant has no effect

On variances and covariances

It's clear from the definitions

- $cov(\mathbf{x}) = E\left\{(\mathbf{x}-\boldsymbol{\mu})(\mathbf{x}-\boldsymbol{\mu})'\right\}$
- $cov(\mathbf{x},\mathbf{y}) = E\left\{(\mathbf{x}-\boldsymbol{\mu}_x)(\mathbf{y}-\boldsymbol{\mu}_y)'\right\}$

That

- $cov(\mathbf{x}+\mathbf{a}) = cov(\mathbf{x})$
- $cov(\mathbf{x}+\mathbf{a},\mathbf{y}+\mathbf{b}) = cov(\mathbf{x},\mathbf{y})$

For example, $E(\mathbf{x}+\mathbf{a}) = \boldsymbol{\mu}+\mathbf{a}$, so

$$\begin{aligned} cov(\mathbf{x}+\mathbf{a}) &= E\left\{(\mathbf{x}+\mathbf{a}-(\boldsymbol{\mu}+\mathbf{a}))(\mathbf{x}+\mathbf{a}-(\boldsymbol{\mu}+\mathbf{a}))'\right\} \\ &= E\left\{(\mathbf{x}-\boldsymbol{\mu})(\mathbf{x}-\boldsymbol{\mu})'\right\} \\ &= cov(\mathbf{x}) \end{aligned}$$

Here's a useful formula

Let $E(\mathbf{y}) = \boldsymbol{\mu}$, $cov(\mathbf{y}) = \boldsymbol{\Sigma}$, and let $\mathbf{A}$ and $\mathbf{B}$ be matrices of constants. Then

$$cov(\mathbf{Ay}, \mathbf{By}) = \mathbf{A\Sigma B}'.$$

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