The Multivariate Normal Distribution

STA 302 Fall 2020

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Overview

1. Joint Moment-generating Functions
2. Definition of the Multivariate Normal Distribution
3. Properties of the Multivariate Normal
4. \( \chi^2 \) and \( t \) Distributions
Joint moment-generating function

Of a $p$-dimensional random vector $x$

- $M_x(t) \overset{def}{=} E \left( e^{t'x} \right)$  
  Compare $M_x(t) = E(e^{xt})$.

- For example,

$$M_{(x_1, x_2, x_3)}(t_1, t_2, t_3) = E \left( e^{x_1 t_1 + x_2 t_2 + x_3 t_3} \right)$$

$$= \int \int \int e^{x_1 t_1 + x_2 t_2 + x_3 t_3} f(x_1, x_2, x_3) \, dx_1 \, dx_2 \, dx_3$$

- Just write $M(t)$ if there is no ambiguity.

Section 4.3 of *Linear models in statistics* has some material on moment-generating functions (optional).
Joint moment-generating functions correspond uniquely to joint probability distributions.

- \( M(t) \) is a function of \( F(x) \).
  - Step One: \( f(x) = \frac{\partial}{\partial x_1} \cdots \frac{\partial}{\partial x_p} F(x) \).
  - For example, \( \frac{\partial}{\partial x_1} \frac{\partial}{\partial x_2} \int_{-\infty}^{x_2} \int_{-\infty}^{x_1} f(y_1, y_2) dy_1 dy_2 \)
  - Step Two: \( M(t) = \int \cdots \int e^{t'x} f(x) dx \)
  - Could write \( M(t) = g(F(x)) \).

Uniqueness says the function \( g \) is one-to-one, so that \( F(x) = g^{-1}(M(t)) \).
$g^{-1}(M(t)) = F(x)$

A two-variable example

\[
g^{-1} \left( \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{x_1 t_1 + x_2 t_2} f(x_1, x_2) \, dx_1 \, dx_2 \right) = \int_{-\infty}^{x_2} \int_{-\infty}^{x_1} f(y_1, y_2) \, dy_1 \, dy_2
\]
Theorem

Two random vectors $\mathbf{x}_1$ and $\mathbf{x}_2$ are independent if and only if the moment-generating function of their joint distribution is the product of their moment-generating functions.
Proof

Two random vectors are independent if and only if the moment-generating function of their joint distribution is the product of their moment-generating functions.

Independence therefore the MGFs factor is an exercise.

\[
M_{x_1,x_2}(t_1, t_2) = M_{x_1}(t_1)M_{x_2}(t_2)
\]
\[
= \left( \int_{-\infty}^{\infty} e^{x_1 t_1} f_{x_1}(x_1) \, dx_1 \right) \left( \int_{-\infty}^{\infty} e^{x_2 t_2} f_{x_2}(x_2) \, dx_2 \right)
\]
\[
= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{x_1 t_1 + x_2 t_2} f_{x_1}(x_1) f_{x_2}(x_2) \, dx_1 \, dx_2
\]
Proof continued

Have \( M_{x_1,x_2}(t_1, t_2) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{x_1 t_1 + x_2 t_2} f_{x_1}(x_1) f_{x_2}(x_2) \, dx_1 dx_2. \)

Using \( F(x) = g^{-1}(M(t)) \),

\[
F(x_1, x_2) = g^{-1} \left( \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{x_1 t_1 + x_2 t_2} f_{x_1}(x_1) f_{x_2}(x_2) \, dx_1 dx_2 \right)
\]

\[
= \int_{-\infty}^{x_2} \int_{-\infty}^{x_1} f_{x_1}(y_1) f_{x_2}(y_2) \, dy_1 dy_2
\]

\[
= \int_{-\infty}^{x_2} f_{x_2}(y_2) \left( \int_{-\infty}^{x_1} f_{x_1}(y_1) \, dy_1 \right) \, dy_2
\]

\[
= \int_{-\infty}^{x_2} f_{x_2}(y_2) F_{x_1}(x_1) \, dy_2
\]

\[
= F_{x_1}(x_1) \int_{-\infty}^{x_2} f_{x_2}(y_2) \, dy_2
\]

\[
= F_{x_1}(x_1) F_{x_2}(x_2)
\]

So that \( x_1 \) and \( x_2 \) are independent. ■
A helpful distinction

- If $x_1$ and $x_2$ are independent,

$$M_{x_1+x_2}(t) = M_{x_1}(t)M_{x_2}(t)$$

- $x_1$ and $x_2$ are independent if and only if

$$M_{x_1,x_2}(t_1, t_2) = M_{x_1}(t_1)M_{x_2}(t_2)$$
Theorem: Functions of independent random vectors are independent

Show $x_1$ and $x_2$ independent implies that $y_1 = g_1(x_1)$ and $y_2 = g_2(x_2)$ are independent.

Let $y = \left( \frac{y_1}{y_2} \right) = \left( \frac{g_1(x_1)}{g_2(x_2)} \right)$ and $t = \left( \frac{t_1}{t_2} \right)$. Then

$$M_y(t) = E\left( e^{t'y} \right) = E \left( e^{(t'_1|t'_2)\left( \frac{y_1}{y_2} \right)} \right)$$

$$= E\left( e^{t'_1y_1+t'_2y_2} \right) = E\left( e^{t'_1y_1}e^{t'_2y_2} \right) = E\left( e^{t'_1g_1(x_1)}e^{t'_2g_2(x_2)} \right)$$

$$= \int \int e^{t'_1g_1(x_1)}e^{t'_2g_2(x_2)}f_{x_1}(x_1)f_{x_2}(x_2) \, dx_1 \, dx_2$$

$$= \int e^{t'_2g_2(x_2)}f_{x_2}(x_2) \left( \int e^{t'_1g_1(x_1)}f_{x_1}(x_1) \, dx_1 \right) \, dx_2$$

$$= \int e^{t'_2g_2(x_2)}f_{x_2}(x_2)M_{g_1(x_1)}(t_1) \, dx_2$$

$$= M_{g_1(x_1)}(t_1)M_{g_2(x_2)}(t_2) = M_{y_1}(t_1)M_{y_2}(t_2)$$

So $y_1$ and $y_2$ are independent. ■
$M_{Ax}(t) = M_x(A't)$

Analogue of $M_{ax}(t) = M_x(at)$

Recalling $M_x(t) \overset{def}{=} E(e^{t'x})$,

\[
M_{Ax}(t) = E(e^{t'Ax}) \\
= E(e^{(A't)'x}) \\
= M_x(A't)
\]

Note that $t$ is the same length as $y = Ax$: The number of rows in $A$. 
Joint Moment-generating Functions

\[ M_{x+c}(t) = e^{t'c} M_x(t) \]

Analogue of \( M_{x+c}(t) = e^{ct} M_x(t) \)

\[
M_{x+c}(t) = E \left( e^{t'(x+c)} \right) \\
= E \left( e^{t'x + t'c} \right) \\
= e^{t'c} E \left( e^{t'x} \right) \\
= e^{t'c} M_x(t)
\]
Not in the text.
Distributions may be defined in terms of moment-generating functions.

Build up the multivariate normal from univariate normals.

- If \( y \sim N(\mu, \sigma^2) \), then \( M_y(t) = e^{\mu t + \frac{1}{2} \sigma^2 t^2} \)
- Moment-generating functions correspond uniquely to probability distributions.
- So define a normal random variable with expected value \( \mu \) and variance \( \sigma^2 \) as a random variable with moment-generating function \( e^{\mu t + \frac{1}{2} \sigma^2 t^2} \).
- This has one surprising consequence ...
Degenerate random variables

A *degenerate* random variable has all the probability concentrated at a single value, say $Pr\{y = y_0\} = 1$. Then

$$M_y(t) = E(e^{yt})$$

$$= \sum_{\{y: p(y)>0\}} e^{yt}p(y)$$

$$= e^{y_0t} \cdot p(y_0)$$

$$= e^{y_0t} \cdot 1$$

$$= e^{y_0t}$$
If $Pr\{y = y_0\} = 1$, then $M_y(t) = e^{y_0 t}$

- This is of the form $e^{\mu t + \frac{1}{2} \sigma^2 t^2}$ with $\mu = y_0$ and $\sigma^2 = 0$.
- So $y \sim N(y_0, 0)$.
- That is, degenerate random variables are “normal” with variance zero.
- Call them singular normals.
- This will be surprisingly handy later.
Independent standard normals

Let \( z_1, \ldots, z_p \stackrel{i.i.d.}{\sim} N(0, 1) \).

\[
\mathbf{z} = \begin{pmatrix}
    z_1 \\
    \vdots \\
    z_p
\end{pmatrix}
\]

\[
E(\mathbf{z}) = \mathbf{0} \quad \text{and} \quad \text{cov}(\mathbf{z}) = \mathbf{I}_p
\]
Moment-generating function of $z$

Using $M_x(t) = e^{\mu t + \frac{1}{2} \sigma^2 t^2}$

$$M_z(t) = \prod_{j=1}^{p} M_{z_j}(t_j)$$

$$= \prod_{j=1}^{p} e^{\frac{1}{2} t_j^2}$$

$$= e^{\frac{1}{2} \sum_{j=1}^{p} t_j^2}$$

$$= e^{\frac{1}{2} t' t}$$
Transform $\mathbf{z}$ to get a general multivariate normal

Remember: A non-negative definite means $\mathbf{v}'\mathbf{A}\mathbf{v} \geq 0$

Let $\Sigma$ be a $p \times p$ symmetric non-negative definite matrix and $\mu \in \mathbb{R}^p$. Let $\mathbf{y} = \Sigma^{1/2}\mathbf{z} + \mu$.

- The elements of $\mathbf{y}$ are linear combinations of independent standard normals.
- Linear combinations of independent normals are normal.
- $\mathbf{y}$ has a multivariate distribution.
- We’d like to call $\mathbf{y}$ a multivariate normal.
Moment-generating function of $y = \Sigma^{1/2}z + \mu$

Remember: $M_{Ax}(t) = M_{x}(A' t)$ and $M_{x+c}(t) = e^{t'c}M_{x}(t)$ and $M_{z}(t) = e^{\frac{1}{2}t'^{t}}$

\[
M_{y}(t) = M_{\Sigma^{1/2}z+\mu}(t) \\
= e^{t'\mu}M_{\Sigma^{1/2}z}(t) \\
= e^{t'\mu}M_{z}(\Sigma^{1/2}t) \\
= e^{t'\mu}e^{\frac{1}{2}(\Sigma^{1/2}t)'(\Sigma^{1/2}t)} \\
= e^{t'\mu}e^{\frac{1}{2}t'\Sigma^{1/2}\Sigma^{1/2}t} \\
= e^{t'\mu}e^{\frac{1}{2}t'\Sigma t} \\
= e^{t'\mu+\frac{1}{2}t'\Sigma t}
\]

So define a multivariate normal random variable $y$ as one with moment-generating function $M_{y}(t) = e^{t'\mu+\frac{1}{2}t'\Sigma t}$. 
Compare univariate and multivariate normal moment-generating functions

Univariate \[ M_y(t) = e^{\mu t + \frac{1}{2} \sigma^2 t^2} \]

Multivariate \[ M_y(t) = e^{t'\mu + \frac{1}{2} t'\Sigma t} \]

So the univariate normal is a special case of the multivariate normal with \( p = 1 \).
Mean and covariance matrix
For a univariate normal, \( E(y) = \mu \) and \( Var(y) = \sigma^2 \)

Recall \( y = \Sigma^{1/2}z + \mu \).

\[
\begin{align*}
E(y) & = \mu \\
cov(y) & = \Sigma^{1/2}cov(z)\Sigma^{1/2'} \\
& = \Sigma^{1/2}I\Sigma^{1/2} \\
& = \Sigma
\end{align*}
\]

We will say \( y \) is multivariate normal with expected value \( \mu \) and variance-covariance matrix \( \Sigma \), and write \( y \sim N_p(\mu, \Sigma) \).

Note that because \( M_y(t) = e^{t'\mu + \frac{1}{2}t'\Sigma t} \), \( \mu \) and \( \Sigma \) completely determine the distribution.
Probability density function of $\mathbf{y} \sim N_p(\mu, \Sigma)$

Remember, $\Sigma$ is only positive semi-definite.

It is easy to write down the density of $\mathbf{z} \sim N_p(\mathbf{0}, I)$ as a product of standard normals.

If $\Sigma$ is strictly positive definite (and not otherwise), the density of $\mathbf{y} = \Sigma^{1/2} \mathbf{z} + \mu$ can be obtained using the Jacobian Theorem as

$$f(\mathbf{y}) = \frac{1}{|\Sigma|^{1/2} (2\pi)^{p/2}} \exp \left\{ -\frac{1}{2} (\mathbf{y} - \mu)' \Sigma^{-1} (\mathbf{y} - \mu) \right\}$$

This is usually how the multivariate normal is defined.
Bivariate Normal Density \((p = 2)\)
Properties of the Multivariate Normal

\( \Sigma \) positive definite?

- Positive definite means that for any non-zero \( p \times 1 \) vector \( \mathbf{a} \), we have \( \mathbf{a}' \Sigma \mathbf{a} > 0 \).

- Since the one-dimensional random variable \( w = \sum_{i=1}^{p} a_i y_i \) may be written as \( w = \mathbf{a}' \mathbf{y} \) and \( \text{Var}(w) = \text{cov}(\mathbf{a}' \mathbf{y}) = \mathbf{a}' \Sigma \mathbf{a} \), it is natural to require that \( \Sigma \) be positive definite.

- All it means is that every non-zero linear combination of \( \mathbf{y} \) values has a positive variance. Often, this is what you want.
Singular normal: \( \Sigma \) is positive semi-definite.
Or “non-negative definite”

Suppose there is \( a \neq 0 \) with \( a' \Sigma a = 0 \). Let \( w = a'y \).

- Then \( Var(w) = cov(a'y) = a' \Sigma a = 0 \). That is, \( w \) has a degenerate distribution (but it’s still still normal).
- In this case we describe the distribution of \( y \) as a singular multivariate normal.
- Including the singular case saves a lot of extra work in later proofs.
- We will insist that a singular multivariate normal is still multivariate normal, even though it has no density.
Let $y \sim N_p(\mu, \Sigma)$, and $w = Ay$, where $A$ is an $r \times p$ matrix.

\[
M_w(t) = M_{Ay}(t) = M_y(A't) = e^{(A't)\mu + \frac{1}{2}(A't)'\Sigma(A't)}
\]

\[
= e^{t'(A\mu) + \frac{1}{2}t'(A\Sigma A')t}
\]

Recognize moment-generating function and conclude

\[
w \sim N_r(A\mu, A\Sigma A')
\]
Exercise
Use moment-generating functions, of course.

Let \( y \sim N_p(\mu, \Sigma) \).

Show \( y + c \sim N_p(\mu + c, \Sigma) \).
Zero covariance implies independence for the multivariate normal.

- Independence always implies zero covariance.
- For the multivariate normal, zero covariance also implies independence.
- The multivariate normal is the only continuous distribution with this property.
Show zero covariance implies independence

By showing \( M_y(t) = M_{y_1}(t_1)M_{y_2}(t_2) \)

Let \( y \sim N(\mu, \Sigma) \), with

\[
\begin{align*}
    y &= \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} & \mu &= \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} & \Sigma &= \begin{pmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{pmatrix} & t &= \begin{pmatrix} t_1 \\ t_2 \end{pmatrix}
\end{align*}
\]

\[
M_y(t) = E \left( e^{t'y} \right)
\]

\[
= E \left( e^{t_1 y_1} e^{t_2 y_2} \right)
\]

\[
= E \left( e^{t_1 y_1} \right) E \left( e^{t_2 y_2} \right)
\]

\[
= E \left( e^{t_1 y_1} \right) E \left( e^{t_2 y_2} \right)
\]

\[
= \ldots
\]
Continuing the calculation: \( M_y(t) = e^{t'\mu + \frac{1}{2}t'\Sigma t} \)

\[ y = \left( \begin{array}{c} y_1 \\ y_2 \end{array} \right), \quad \mu = \left( \begin{array}{c} \mu_1 \\ \mu_2 \end{array} \right), \quad \Sigma = \left( \begin{array}{cc} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{array} \right), \quad t = \left( \begin{array}{c} t_1 \\ t_2 \end{array} \right) \]

\[
M_y(t) = E \left( e \left( \begin{array}{c} t_1 \\ t_2 \end{array} \right)' y \right)
\]

\[
= \exp \left\{ (t_1'|t_2') \left( \frac{\mu_1}{\mu_2} \right) \right\} \exp \left\{ \frac{1}{2} (t_1'|t_2') \left( \begin{array}{cc} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{array} \right) \left( \begin{array}{c} t_1 \\ t_2 \end{array} \right) \right\}
\]

\[
= e^{t_1'\mu_1 + t_2'\mu_2} \exp \left\{ \frac{1}{2} (t_1' \Sigma_1 | t_2' \Sigma_2) \left( \begin{array}{c} t_1 \\ t_2 \end{array} \right) \right\}
\]

\[
= e^{t_1'\mu_1 + t_2'\mu_2} \exp \left\{ \frac{1}{2} (t_1' \Sigma_1 t_1 + t_2' \Sigma_2 t_2) \right\}
\]

\[
= e^{t_1'\mu_1} \ e^{t_2'\mu_2} \ e^{\frac{1}{2} (t_1' \Sigma_1 t_1)} \ e^{\frac{1}{2} (t_2' \Sigma_2 t_2)}
\]

\[
= e^{t_1'\mu_1 + \frac{1}{2} (t_1' \Sigma_1 t_1)} \ e^{t_2'\mu_2 + \frac{1}{2} (t_2' \Sigma_2 t_2)}
\]

\[
= M_{y_1}(t_1) M_{y_2}(t_2)
\]

So \( y_1 \) and \( y_2 \) are independent. ■
Let $y_1 \sim N(1, 2)$, $y_2 \sim N(2, 4)$ and $y_3 \sim N(6, 3)$ be independent, with $w_1 = y_1 + y_2$ and $w_2 = y_2 + y_3$. Find the joint distribution of $w_1$ and $w_2$.

\[
\begin{pmatrix}
w_1 \\
w_2
\end{pmatrix} = \begin{pmatrix}1 & 1 & 0 \\0 & 1 & 1\end{pmatrix} \begin{pmatrix}y_1 \\y_2 \\y_3\end{pmatrix}
\]

\[w = Ay \sim N(A\mu, A\Sigma A')\]
\[ w = Ay \sim N(A\mu, A\Sigma A') \]

\[ y_1 \sim N(1, 2), \ y_2 \sim N(2, 4) \text{ and } y_3 \sim N(6, 3) \text{ are independent} \]

\[ A\mu = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 6 \end{pmatrix} = \begin{pmatrix} 3 \\ 8 \end{pmatrix} \]

\[ A\Sigma A' = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 6 & 4 \\ 4 & 7 \end{pmatrix} \]
Marginal distributions are multivariate normal \( \mathbf{y} \sim N_p(\mu, \Sigma) \), so \( \mathbf{w} = \mathbf{A}\mathbf{y} \sim N(\mathbf{A}\mu, \mathbf{A}\Sigma\mathbf{A}' ) \)

Find the distribution of

\[
\begin{pmatrix}
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
y_1 \\
y_2 \\
y_3 \\
y_4
\end{pmatrix} =
\begin{pmatrix}
y_2 \\
y_4
\end{pmatrix}
\]

Bivariate normal. The expected value is easy.
Covariance matrix of \( Ay \)

\[
\text{cov}(Ay) = A\Sigma A'
\]

\[
= \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \sigma_1^2 & \sigma_{1,2} & \sigma_{1,3} & \sigma_{1,4} \\ \sigma_{1,2} & \sigma_2^2 & \sigma_{2,3} & \sigma_{2,4} \\ \sigma_{1,3} & \sigma_{2,3} & \sigma_3^2 & \sigma_{3,4} \\ \sigma_{1,4} & \sigma_{2,4} & \sigma_{3,4} & \sigma_4^2 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}
\]

\[
= \begin{pmatrix} \sigma_{1,2} & \sigma_2^2 & \sigma_{2,3} & \sigma_{2,4} \\ \sigma_{1,4} & \sigma_{2,4} & \sigma_{3,4} & \sigma_4^2 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}
\]

\[
= \begin{pmatrix} \sigma_2^2 & \sigma_{2,4} \\ \sigma_{2,4} & \sigma_4^2 \end{pmatrix}
\]

Marginal distributions of a multivariate normal are multivariate normal, with the original means, variances and covariances.
Summary

- If $\mathbf{c}$ is a vector of constants, $\mathbf{x} + \mathbf{c} \sim N(\mathbf{c} + \mu, \Sigma)$.
- If $\mathbf{A}$ is a matrix of constants, $\mathbf{A}\mathbf{x} \sim N(A\mu, A\Sigma A')$.
- Linear combinations of multivariate normals are multivariate normal.
- All the marginals (dimension less than $p$) of $\mathbf{x}$ are (multivariate) normal, but it is possible in theory to have a collection of univariate normals whose joint distribution is not multivariate normal.
- For the multivariate normal, zero covariance implies independence. The multivariate normal is the only continuous distribution with this property.
Need the multivariate normal for this.
Showing \( w = (x - \mu)' \Sigma^{-1} (x - \mu) \sim \chi^2(p) \)

\( \Sigma \) has to be positive definite this time

\[
\begin{align*}
x & \sim N(\mu, \Sigma) \\
y = x - \mu & \sim N(0, \Sigma) \\
z = \Sigma^{-\frac{1}{2}} y & \sim N\left(0, \Sigma^{-\frac{1}{2}} \Sigma \Sigma^{-\frac{1}{2}}\right) \\
& = N\left(0, \Sigma^{-\frac{1}{2}} \Sigma^{\frac{1}{2}} \Sigma^{\frac{1}{2}} \Sigma^{-\frac{1}{2}}\right) \\
& = N(0, I_p)
\end{align*}
\]

So \( z \) is a vector of \( p \) independent standard normals, and

\[
\begin{align*}
w &= (x - \mu)' \Sigma^{-1} (x - \mu) \\
&= y' \Sigma^{-1} y = y' \Sigma^{-1/2} \Sigma^{-1/2} y \\
&= (\Sigma^{-\frac{1}{2}} y)' (\Sigma^{-\frac{1}{2}} y) \\
&= z' z \\
&= \sum_{j=1}^{p} z_i^2 \sim \chi^2(p) \quad \blacksquare
\end{align*}
\]
Show $\bar{x}$ and $s^2$ independent

$x_1, \ldots, x_n \overset{i.i.d.}{\sim} N(\mu, \sigma^2)$

$$x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \sim N(\mu j, \sigma^2 I)$$

$$y = \begin{pmatrix} x_1 - \bar{x} \\ \vdots \\ x_n - \bar{x} \\ \bar{x} \end{pmatrix} = Ax$$

Note $A$ is $(n + 1) \times n$, so $cov(Ax) = \sigma^2 AA'$ is $(n + 1) \times (n + 1)$, singular.
The argument

\[ y = A \mathbf{x} = \begin{pmatrix} x_1 - \bar{x} \\ \vdots \\ x_n - \bar{x} \\ \bar{x} \\ \bar{x} \end{pmatrix} = \begin{pmatrix} y_2 \\ \bar{x} \end{pmatrix} \]

- \( y \) is multivariate normal because \( \mathbf{x} \) is multivariate normal.
- \( \text{Cov}(\bar{x}, x_j - \bar{x}) = 0 \) (Exercise)
- So \( \bar{x} \) and \( y_2 \) are independent.
- So \( \bar{x} \) and \( s^2 = g(y_2) \) are independent.
Leads to the $t$ distribution

If

- $z \sim N(0, 1)$ and
- $y \sim \chi^2(\nu)$ and
- $z$ and $y$ are independent, then we say

$$t = \frac{z}{\sqrt{y/\nu}} \sim t(\nu)$$
Random sample from a normal distribution

Let $x_1, \ldots, x_n \overset{i.i.d.}{\sim} N(\mu, \sigma^2)$. Then

- $\frac{\sqrt{n(x - \mu)}}{\sigma} \sim N(0, 1)$ and
- $\frac{(n-1)s^2}{\sigma^2} \sim \chi^2(n - 1)$ and
- These quantities are independent, so

$$
t = \frac{\sqrt{n(x - \mu)} / \sigma}{\sqrt{\frac{(n-1)s^2}{\sigma^2}} / (n - 1)} = \frac{\sqrt{n(x - \mu)}}{s} \sim t(n - 1)$$
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http://www.utstat.toronto.edu/~brunner/oldclass/302f20