Assignment 9

1 (a) $y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 + \beta_4 x_4 + \beta_5 x_5 + \beta_6 x_6 + \beta_7 h_1 + \beta_8 h_2 + \varepsilon$

(b)

	h_1	h_2	$E(y)$
city	0	0	$\beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 + \beta_4 x_4 + \beta_5 x_5 + \beta_6 x_6$
sub	1	0	$\beta_0 + \beta_7 +$ "
country	0	1	$\beta_0 + \beta_8 +$ "

I made city the reference category, but that's just because I'm a city boy.

(c) i. $H_0: \beta_7 = \beta_8 = 0$

$$\underbrace{\begin{pmatrix} 0&0&0&0&0&0&0&1&0 \\ 0&0&0&0&0&0&0&0&1 \end{pmatrix}}_{C} \beta = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\beta = t$$

Reduced $E(y \mid \underset{\sim}{x}) = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 + \beta_4 x_4 + \beta_5 x_5 + \beta_6 x_6$

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ii. $H_0: \beta_1 = \beta_2 = 0$

$$\underbrace{\begin{pmatrix} 0&1&0&0&0&0&0&0&0 \\ 0&0&1&0&0&0&0&0&0 \end{pmatrix}}_{C} \beta = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\beta = t$$

Reduced $E(y \mid \underset{\sim}{x}) =$

$\beta_0 + \beta_3 x_3 + \beta_4 x_4 + \beta_5 x_5 + \beta_6 x_6 + \beta_7 h_1 + \beta_8 h_2$

(1 c iii)

$H_0: \beta_6 = 0$

$$\underbrace{(0\ 0\ 0\ 0\ 0\ 0\ 1\ 0\ 0)}_{C}\ \underbrace{\beta}_{\beta} = \underbrace{(0)}_{t}$$

Reduced $E(y \mid \underset{\sim}{x}) = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 + \beta_4 x_4 + \beta_5 x_5 + \beta_7 h_1 + \beta_8 h_2$

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iv. $H_0: \beta_1 = \beta_2$

$$\underbrace{(0\ 1\ -1\ 0\ 0\ 0\ 0\ 0\ 0)}_{C}\ \underbrace{\beta}_{\beta} = \underbrace{(0)}_{t}$$

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v. $H_0: \beta_4 = \beta_5 = \beta_6 = \beta_7 = \beta_8 = 0$

$$\underbrace{\begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}}_{C} \underbrace{\beta}_{\beta} = \underbrace{\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}}_{t}$$

Reduced $E(y \mid \underset{\sim}{x}) = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3$

(1c vi) $H_0: \beta_4 = \beta_5$

$$\underset{C}{(0\ 0\ 0\ 0\ 1\ {-1}\ 0\ 0\ 0)}\beta = (0)$$
$$C\beta = t$$

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vii. $H_0: \beta_2 = 0$

$$\underset{C}{(0\ 0\ 1\ 0\ 0\ 0\ 0\ 0\ 0)}\beta = (0)$$
$$C\beta = t$$

Reduced $E(y|\underset{\sim}{x}) = \beta_0 + \beta_1 x_1 + \beta_3 x_3 + \beta_4 x_4 + \beta_5 x_5 + \beta_6 x_6 + \beta_7 h_1 + \beta_8 h_2$

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viii) $H_0: \beta_7 = 0$

$$\underset{C}{(0\ 0\ 0\ 0\ 0\ 0\ 0\ 1\ 0)}\beta = (0)$$
$$C\beta = t$$

Reduced $E(y|\underset{\sim}{x}) = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 + \beta_4 x_4 + \beta_5 x_5 + \beta_6 x_6 + \beta_8 x_8$

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ix. $H_0: \beta_1 = \beta_2 = \beta_3 = \beta_4 = \beta_5 = \beta_6 = 0$

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}\beta = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$E(y|\underset{\sim}{x}) = \beta_0 + \beta_7 h_1 + \beta_8 h_2$

(1d) i. $E(y|\underset{\sim}{x}) = \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 + \beta_4 x_4 + \beta_5 x_5 + \beta_6 x_6$
$+ \beta_7 h_1 + \beta_8 h_2 + \beta_9 h_3$

ii.

	h_1	h_2	h_3	$E(y\mid\underline{x})$
city	1	0	0	$\beta_7 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 + \beta_4 x_4 + \beta_5 x_5 + \beta_0 x_6$
suburbs	0	1	0	$\beta_8 +$ "
country	0	0	1	$\beta_9 +$ "

iii. $H_0: \beta_7 = \beta_8 = \beta_9$

iv. $H_0: \beta_7 = \beta_8$

(2) The first 3 rows & columns of X are a 3x3 identity matrix, so

$$A = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

(3) The sum of indicators for the first categorical variable will add to one, and so will the sum of indicators for the second categorical variable. Subtract one from the other & you get zero, for every row of X. So, the columns of X are linearly dependent.

4 (a) $\hat{\alpha} = (W'W)^{-1}W'y$

$= ((XA)'XA)^{-1}(XA)'y = (A'X'XA)^{-1}A'X'y$

$= A^{-1}(X'X)^{-1}\underbrace{(A')^{-1}A'}_{I}X'y = A^{-1}(X'X)^{-1}X'y$

$= A^{-1}\hat{\beta}$

(b) $\hat{y} = W\hat{\alpha} = XAA^{-1}\hat{\beta} = X\hat{\beta}$

Same $\hat{y}$

(c) $H_0: C\beta = t \iff CAA^{-1}\beta = t$

$\iff CA\alpha = t \iff C_2\alpha = t, \quad C_2 = CA$

(d) For $H_0: C_2\alpha = t$,

$F^* = (C_2\hat{\alpha} - t)'(C_2(W'W)^{-1}C_2')^{-1}(C_2\hat{\alpha} - t)/(q\,MSE)$

$= (C\underbrace{AA^{-1}}_{I}\hat{\beta} - t)(CA((XA)'XA)^{-1}(CA)')^{-1}(CAA^{-1}\hat{\beta} - t)/(q\,MSE)$

$= (C\hat{\beta} - t)'(CA(A'X'XA)^{-1}A'C')^{-1}(C\hat{\beta} - t)/(q\,MSE)$

$= (C\hat{\beta} - t)'(C\underbrace{AA^{-1}}_{I}(X'X)^{-1}\underbrace{(A')^{-1}A'}_{I}C')^{-1}(C\hat{\beta} - t)/(q\,MSE)$

$= F^*$ for $H_0: C\beta = t$. Note MSE is the same because $\hat{y}$ is the same.

5 (a) i. $\sum_{i=1}^{n} y_i^2 = \sum_{i=1}^{n} (y_i - \hat{y}_i + \hat{y}_i)^2$

$= \sum_{i=1}^{n} (y_i - \hat{y}_i)^2 + 2\sum_{i=1}^{n} \hat{\varepsilon}_i \hat{y}_i + \sum_{i=1}^{n} \hat{y}_i^2$

$= SSE + \hat{y}'\hat{\varepsilon} + \sum_{i=1}^{n} \hat{y}_i^2$

$= SSE + (X\hat{\beta})'\hat{\varepsilon} + \sum_{i=1}^{n} \hat{y}_i^2 = SSE + \hat{\beta}'\underbrace{X'\hat{\varepsilon}}_{0} + \sum_{i=1}^{n} \hat{y}_i^2$

$= SSE + \sum_{i=1}^{n} \hat{y}_i^2$ ∎

ii. $R^2 = \dfrac{\sum_{i=1}^{n} \hat{y}_i^2}{\sum_{i=1}^{n} y_i^2}$

(b) $\sum_{i=1}^{n} \hat{\varepsilon}_i = j'\hat{\varepsilon} = (Xv)'\hat{\varepsilon} = v'\underbrace{X'\hat{\varepsilon}}$

$= v'\underset{\sim}{0} = 0$.

6 (a) $E(y \mid \underset{\sim}{x}) = \beta_1 r_1 + \beta_2 r_2 + \beta_3 r_3 + \beta_4 r_4 + \beta_5 x$

(b)

	r_1	r_2	r_3	r_4	$E(y \mid x)$
North Central	1	0	0	0	$\beta_1 + \beta_5 x$
North East	0	1	0	0	$\beta_2 + \beta_5 x$
South	0	0	1	0	$\beta_3 + \beta_5 x$
West	0	0	0	1	$\beta_4 + \beta_5 x$

(c) i. $H_0: \beta_1 = \beta_2 = \beta_3 = \beta_4$

ii. $H_0: \beta_1 = \beta_2$

iii. $H_0: \beta_2 = \beta_4$

iv. $H_0: \beta_3 = \frac{1}{3}(\beta_1 + \beta_2 + \beta_4)$

v. $H_0: \beta_1 + \beta_2 = \beta_3 + \beta_4$

vi. $H_0: \beta_5 = 0$

(d) Both questions imply influence of the predictor variable on the predicted variable. See the lecture slides on correlation - causation.

(6e) $E(y|x) = \beta_0 + \beta_1 x + \beta_2 r_2 + \beta_3 r_3 + \beta_4 r_4$

$+ \beta_5 x r_2 + \beta_6 x r_3 + \beta_7 x r_4$

(f)

	r_2	r_3	r_4	$E(y \mid x)$
North Central	0	0	0	$\beta_0 + \beta_1 x$
Northeast	1	0	0	$(\beta_0 + \beta_2) + (\beta_1 + \beta_5) x$
South	0	1	0	$(\beta_0 + \beta_3) + (\beta_1 + \beta_6) x$
West	0	0	1	$(\beta_0 + \beta_4) + (\beta_1 + \beta_7) x$

(g) i. $H_0: \beta_5 = \beta_6 = \beta_7 = 0$

ii. $H_0: \beta_5 = \beta_6 = \beta_7 = 0$

iii. $H_0: \beta_5 = \beta_6 = \beta_7 = 0$

iv. $H_0: \beta_5 = \beta_6 = \beta_7 = 0$

v. $H_0: \beta_5 = 0$

vi. $H_0: \beta_6 = 0$

vii. $H_0: \beta_7 = 0$

viii. $H_0: \beta_5 = \beta_6$

ix. $H_0: \beta_5 = \beta_7$

x. $H_0: \beta_6 = \beta_7$

xi. $H_0: \beta_1 + \beta_6 = 0$

xii. $H_0: \beta_1 + \beta_5 = 0, \ \beta_1 + \beta_6 = 0$

7 (a)

	r_2	r_3	r_4	$E(y \mid x)$
North Central	0	0	0	β_0
Northeast	1	0	0	$(\beta_0+\beta_2) + \beta_5 x$
South	0	1	0	$(\beta_0+\beta_3) + \beta_6 x$
West	0	0	1	$(\beta_0+\beta_4) + \beta_7 x$

This weird model is saying that in the north central region, crime rate does not depend on income at all. In the other 3 regions, slopes as well as intercepts might be different.

(b)

	r_2	r_3	r_4	
NC	0	0	0	$\beta_0 + \beta_1 x$
NE	1	0	0	$\beta_0 + (\beta_1+\beta_5)x$
S	0	1	0	$\beta_0 + (\beta_1+\beta_6)x$
W	0	0	1	$\beta_0 + (\beta_1+\beta_7)x$

The slopes might be different, but the intercepts are all the same.

(7c)

$$E(y|x) = \beta_1 r_1 + \beta_2 r_2 + \beta_3 r_3 + \beta_4 r_4 + \beta_5 x + \beta_6 x r_1 + \beta_7 x r_2 + \beta_8 x r_3 + \beta_9 x r_4$$

i. The model with an intercept and 3 dummy variables has eight regression coefficients. This one has nine. They can't possibly be one-to-one.

ii. $x r_1 + x r_2 + x r_3 + x r_4 = x$, so the columns of the X matrix are linearly dependent.

iii. Drop $\beta_9 x r_4$

	r_1	r_2	r_3	r_4	$E(y\|x)$
NC	1	0	0	0	$\beta_1 + (\beta_5 + \beta_6)x$
NE	0	1	0	0	$\beta_2 + (\beta_5 + \beta_7)x$
S	0	0	1	0	$\beta_3 + (\beta_5 + \beta_8)x$
W	0	0	0	1	$\beta_4 + \beta_8 x$

This is okay. West is the reference category.

(7c iv)

	R_1	R_2	R_3	R_4	$E(y\mid x) = \beta_1 R_1 + \beta_2 R_2 + \beta_3 R_3 + \beta_4 R_4 + \beta_5 R_1 x + \beta_6 R_2 x + \beta_7 R_3 x + \beta_8 R_4 x$
NC	1	0	0	0	$\beta_1 + \beta_5 x$
NE	0	1	0	0	$\beta_2 + \beta_6 x$
S	0	0	1	0	$\beta_3 + \beta_7 x$
W	0	0	0	1	$\beta_4 + \beta_8 x$

It works!

8 a

i) $E(y|\underset{\sim}{x}) = \beta_0 + \beta_1 d_1 + \beta_2 d_2$

Dummy variable for software one

Dummy variable for software two

ii) $R^2 = 0.08176$

iii) $H_0: \beta_1 = \beta_2 = 0$

iv) $F = 1.469$

v) $p = 0.2448$

vi) No.

vii) No.

viii) One vs Two : $p = 0.669$

One vs Three: $p = 0.108$

Two vs Three $p = 0.231$

ix) Ignoring sales last quarter, there is no evidence that software package has any effect on sales this quarter

8b

i) $E(y \mid \underset{\sim}{x}) = \beta_0 + \beta_1 x + \beta_2 d_1 + \beta_2 d_2$

sales last quarter

same as 8a

ii) $H_0: \beta_2 = \beta_3 = 0$

iii) $F = 0.2422$

iv) $p = 0.7863$

v) No.

vi) No.

vii) 0.015

viii) One vs two: $p = 0.7984$
One vs three: $p = 0.6748$
Two vs three: $p = 0.4938$

ix) Allowing for sales last quarter, there is no evidence that software package affects sales this quarter.

8c

i) $E(y|x) = \beta_0 + \beta_1 x + \beta_2 d_1 + \beta_3 d_2 + \beta_4 x d_1 + \beta_5 x d_2$

ii) You really need to make a table, or look at a table in the lecture slides. See (v) below.

$H_0: \beta_4 = \beta_5 = 0$

iii) $H_0: \beta_4 = \beta_5 = 0$

iv) A. $F = 10.305$

B. $p = 0.000392$

C. Yes.

D. Yes.

v)

Software Package	d_1	d_2	$E(y\|x)$
1	1	0	$\beta_0 + \beta_2 + (\beta_1 + \beta_4) x$
2	0	1	$\beta_0 + \beta_3 + (\beta_1 + \beta_5) x$
3	0	0	$\beta_0 + \beta_1 x$

Software package	Estimated slope
1	2.212208
2	0.3542812
3	1.588336

(8c vi)

A. $H_0: \beta_1 + \beta_5 = 0$

B. $F = 1.252$

C. NO.

D. No.

vii)

A. $p = 0.000089$ or 8

B. $p = 0.11820$

C. $p = 0.00710$

For significance with a Bonferroni correction, need $p < 0.05/3 = 0.0167$, so conclude that the slopes for software packages one and three are greater than for package 2. This directional conclusion comes from part (v).

(9)

(a)

	d_1	d_2	$E(y \mid \underset{\sim}{x})$
A	1	0	$\beta_0+\beta_1 + (\beta_3+\beta_5)x + (\beta_4+\beta_7)x^2$
B	0	1	$\beta_0+\beta_2 + (\beta_3+\beta_6)x + (\beta_4+\beta_8)x^2$
C	0	0	$\beta_0 + \beta_3 x + \beta_4 x^2$

(b) $H_0: \beta_5 = \beta_6 = \beta_7 = \beta_8 = 0$