# Assignment 6

(1) False

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> # Residuals need not add to zero
> x = 1:10; y = 2*x + rnorm(10); y = round(y,2)
> rbind(x,y)
  [,1] [,2] [,3] [,4] [,5] [,6]  [,7]  [,8]  [,9] [,10]
x 1.00 2.00 3.00 4.00 5.00  6.0  7.00  8.00  9.00 10.00
y 3.98 3.65 5.08 7.02 8.22  9.5 13.57 16.45 17.62 20.69
>
> noint = lm(y ~ 0 + x) # Fit a model with no intercept
> sum(residuals(noint))
[1] -0.6157143
```

(2) True; $E(\hat{\varepsilon}) = E(y - \hat{y}) = E(y) - E(\hat{y})$

$$= X\beta - E(Hy) = X\beta - HE(y)$$

$$= X\beta - X\underbrace{(X'X)^{-1}X'X}_{I}\beta = X\beta - X\beta = 0$$

(3) True: Know $X'\hat{\varepsilon} = 0$; and if the model has an intercept, the first column of $X$ is all ones; call it $j$. Then the first element of $X'\hat{\varepsilon}$ is a scalar zero $= j'\hat{\varepsilon} = \sum_{i=1}^{n} \varepsilon_i = 0$

(4)(a) $\hat{\hat{\beta}} = (X'X)^{-1}X'\hat{y} = (X'X)^{-1}X'X\hat{\beta} = \hat{\beta}$

Not surprising. All the points in $\hat{y}$ are already on the best fitting plane.

(b) $\hat{\hat{y}} = X\hat{\hat{\beta}} = X\hat{\beta} = \hat{y}$

Not surprising: $\hat{y} \in \mathcal{V}$ already, and projecting $\hat{y}$ onto $\mathcal{V}$ does not change it.

$\hat{y} = Hy = HHy = H\hat{y}$ (It's the same $H$ because $\hat{\hat{\beta}} = \hat{\beta}$)

5 (a) $\hat{\vec{\beta}} = (X'X)^{-1} X' \hat{\vec{\varepsilon}} = (X'X)^{-1} 0 = 0$

(b) $\hat{\vec{y}} = X \hat{\vec{\beta}} = 0$

The shadow of $\vec{\varepsilon}$ on $V$ = space spanned by columns of $X$ is zero.

6 (a) $M_{Ax}(t) = E(e^{t'Ax}) = E(e^{(A't)'x})$

$= M_x(A't)$

(b) $M_{x+c}(t) = E(e^{t'(x+c)}) = E(e^{t'x + t'c})$

$= E(e^{t'x} e^{t'c}) = e^{t'c} E(e^{t'x})$

$= e^{t'c} M_x(t)$

(7) $M_{\underset{\sim}{x}}(\underset{\sim}{t}) = E(e^{t'x})$

Indep.

$$= \iint e^{(t_1' | t_2')\binom{x_1}{x_2}} \underbrace{f_{x_1}(\underset{\sim}{x}_1)\, f_{x_2}(\underset{\sim}{x}_2)}_{\text{Indep.}}\, d\underset{\sim}{x}_1\, d\underset{\sim}{x}_2$$

These are multiple integrals.

$$= \iint e^{t_1'x_1 + t_2'x_2} f_{x_1}(x_1)\, f_{x_2}(x_2)\, dx_1\, dx_2$$

$$= \iint e^{t_1'x_1} f_{x_1}(x_1)\, dx_1\, e^{t_2'x_2} f_{x_2}(x_2)\, dx_2$$

$$= M_{x_1}(t_1) \int e^{t_2'x_2} f_{x_2}(x_2)\, dx_2$$

$$= M_{x_1}(t_1)\, M_{x_2}(t_2)$$

(8) $M_{g_1,g_2}(t_1, t_2) = E\left(e^{(t_1,t_2)\binom{g_1}{g_2}}\right)$ 4

$$= E\left(e^{(t_1,t_2)\binom{g_1(w_1)}{g_2(w_2)}}\right) = E\left(e^{t_1 g_1(w_1) + t_2 g_2(w_2)}\right)$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{g_1(w_1)t_1} e^{g_2(w_2)t_2} \overbrace{f_{w_1}(w_1) f_{w_2}(w_2)}^{\text{indep}} \, dw_1\, dw_2$$

$$= \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} e^{g_1(w_1)t_1} f_{w_1}(w_1)\, dw_1\right) * e^{g_2(w_2)t_2} f_{w_2}(w_2)\, dw_2$$

$$= E\left(e^{g_1(w_1)t_1}\right)\int_{-\infty}^{\infty} e^{g_2(w_2)t_2} f_{w_2}(w_2)\, dw_2$$

$$= E\left(e^{g_1(w_1)t_1}\right) E\left(e^{g_2(w_2)t_2}\right)$$

$$= M_{g_1(w_1)}(t_1)\, M_{g_2(w_2)}(t_2) = M_{g_1}(t_1)\, M_{g_2}(t_2)$$

Therefore independent.

(9) (a) $M_y(t) = E(e^{yt}) = \sum_{y:\,p(y)>0} e^{yt}\, p(t)$

$= e^{\mu t} \cdot P(y=\mu) = e^{\mu t} \cdot 1 = e^{\mu t}$

(b) $e^{\mu t}$ is of the form $e^{\mu t + \frac{1}{2}\sigma^2 t^2}$, with $\sigma^2 = 0$

(10) Want $Var(y) = 0 = a'\Sigma a$

$= (a_1, a_2)\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$

Noticing that the columns of $\Sigma$ are linearly dependent, look for $a_1$ & $a_2$ satisfying

$\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

That is, $a_1 + 2a_2 = 0$ & $2a_1 + 4a_2 = 0$

$a_1 = 2$ and $a_2 = -1$ will do it.

Then, $M_y(t) = M_{a'x}(\underset{\text{scalar}}{t}) = M_x(at) = M_x\begin{pmatrix} a_1 t \\ a_2 t \end{pmatrix}$

$= e^{(at)'\mu + \frac{1}{2}(at)'\Sigma(at)}$

$= e^{(2t,\,-t)\begin{pmatrix} 1 \\ 5 \end{pmatrix} + \frac{1}{2}(2t,\,-t)\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}\begin{pmatrix} 2t \\ -t \end{pmatrix}}$

$= e^{-3t + \frac{1}{2}t^2 (2,\,-1)\begin{pmatrix} 0 \\ 0 \end{pmatrix}} = e^{-3t}$ //

MGF of Degenerate at $-3$

(11) (a) $E(z) = 0$

(b) $\text{cov}(z) = I_p$

(c) $M_z(t) \overset{\text{ind}}{=} \prod_{j=1}^{p} M_{z_j}(t_j) = \prod_{j=1}^{p} e^{\frac{1}{2} t_j^2}$

$= e^{\frac{1}{2}\sum_{j=1}^{p} t_j^2} = e^{\frac{1}{2} t' t}$

(d) (i) $E(y) = E\left(\Sigma^{\frac{1}{2}} z + \mu\right) = \Sigma^{\frac{1}{2}} E(z) + E(\mu)$

$= 0 + \mu = \mu$

(ii) $\text{Cov}(y) = \text{Cov}\left(\Sigma^{\frac{1}{2}} z + \mu\right) = \text{cov}\left(\Sigma^{\frac{1}{2}} z\right)$

$= \Sigma^{\frac{1}{2}} \text{cov}(z) \Sigma^{\frac{1}{2}\prime} = \Sigma^{\frac{1}{2}} I_p \Sigma^{\frac{1}{2}}$

$= \Sigma$

(iii) $M_y(t) = M_{\Sigma^{\frac{1}{2}} z + \mu}(t) = e^{t'\mu} M_{\Sigma^{\frac{1}{2}} z}(t)$

$= e^{t'\mu} M_z(\Sigma^{1/2\prime} t) = e^{t'\mu} e^{\frac{1}{2}(\Sigma^{1/2\prime} t)' \Sigma^{\frac{1}{2}} t}$

$= e^{t'\mu} e^{\frac{1}{2} t' \Sigma^{\frac{1}{2}} \Sigma^{\frac{1}{2}} t}$

$= e^{t'\mu + \frac{1}{2} t' \Sigma t}$

(12) $M_y(t) = e^{t'\mu + \frac{1}{2}t'\Sigma t}$

$$= \mathrm{Exp}\left\{ (t_1, t_2)\begin{pmatrix}\mu_1\\ \mu_2\end{pmatrix} + \frac{1}{2}(t_1, t_2)\begin{pmatrix}\sigma_1^2 & 0\\ 0 & \sigma_2^2\end{pmatrix}\begin{pmatrix}t_1\\ t_2\end{pmatrix} \right\}$$

$$= \mathrm{Exp}\left\{ \mu_1 t_1 + \mu_2 t_2 + \frac{1}{2}\left(t_1\sigma_1^2, t_2\sigma_2^2\right)\begin{pmatrix}t_1\\ t_2\end{pmatrix} \right\}$$

$$= \mathrm{Exp}\left\{ \mu_1 t_1 + \mu_2 t_2 + \frac{1}{2}\left(\sigma_1^2 t_1^2 + \sigma_2^2 t_2^2\right) \right\}$$

$$= e^{\mu_1 t_1 + \frac{1}{2}\sigma_1^2 t_1^2}\, e^{\mu_2 t_2 + \frac{1}{2}\sigma_2^2 t_2^2}$$

$= M_{y_1}(t_1)\, M_{y_2}(t_2)$ so $y_1$ & $y_2$ are independent.

(13)
$$\begin{pmatrix}x_1\\ x_2\\ x_3\end{pmatrix} \sim N\left(\begin{pmatrix}1\\ 0\\ 6\end{pmatrix}, \begin{pmatrix}1 & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & 1\end{pmatrix}\right),$$

$$\underset{y}{\begin{pmatrix}y_1\\ y_2\end{pmatrix}} = \underset{A}{\begin{pmatrix}1 & 1 & 0\\ 0 & 1 & 1\end{pmatrix}} \underset{x}{\begin{pmatrix}x_1\\ x_2\\ x_3\end{pmatrix}}$$

$y \sim N_2(A\mu, A\Sigma A')$, $A\mu = \begin{pmatrix}1\\ 6\end{pmatrix}$

$$A\Sigma A' = \begin{pmatrix}1 & 1 & 0\\ 0 & 1 & 1\end{pmatrix}\begin{pmatrix}1 & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ 1 & 1\\ 0 & 1\end{pmatrix} = \begin{pmatrix}3 & 2\\ 2 & 3\end{pmatrix}$$

(14)

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$y \quad = \quad A \quad x$$

$$E(y) = A\,E(x) = \begin{pmatrix} \mu_1 + \mu_2 \\ \mu_1 - \mu_2 \end{pmatrix}$$

$$\mathrm{cov}(y) = A\,\mathrm{cov}(x)\,A' = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} \sigma_1^2 & \sigma_2^2 \\ \sigma_1^2 & -\sigma_2^2 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} \sigma_1^2 + \sigma_2^2 & \sigma_1^2 - \sigma_2^2 \\ \sigma_1^2 - \sigma_2^2 & \sigma_1^2 + \sigma_2^2 \end{pmatrix}$$

So $$\underset{\sim}{y} \sim N\left( \begin{pmatrix} \mu_1 + \mu_2 \\ \mu_1 - \mu_2 \end{pmatrix}, \begin{pmatrix} \sigma_1^2 + \sigma_2^2 & \sigma_1^2 - \sigma_2^2 \\ \sigma_1^2 - \sigma_2^2 & \sigma_1^2 + \sigma_2^2 \end{pmatrix} \right)$$

Need $\sigma_1^2 = \sigma_2^2$ for independence.

(15) $M_w(t) = M_{Ay+c}(t) = e^{t'c} M_{Ay}(t)$

$= e^{t'c} M_y(A't) = e^{t'c} e^{(A't)'\mu + \frac{1}{2}(A't)'\Sigma A't}$

$= e^{t'c + t'A\mu + \frac{1}{2}t'A\Sigma A't}$

$= e^{t'(A\mu+c) + \frac{1}{2}t'(A\Sigma A')t}$

MGF of $N_r(A\mu+c, A\Sigma A')$

(16) (a) $y \sim N(a\mu+b, a^2\sigma^2)$

(b) $z \sim N(0,1)$

(c) $y \sim N(n\mu, n\sigma^2)$

(d) $\bar{x} \sim N\left(\mu, \frac{\sigma^2}{n}\right)$

(e) $z \sim N(0,1)$

(f) $y \sim N\left(a_0 + \sum_{i=1}^{n} a_i\mu_i, \sum_{i=1}^{n} a_i^2\sigma_i^2\right)$

(g) $y \sim \chi^2\left(\sum_{i=1}^{n} \nu_i\right)$

(h) $y \sim \chi^2(1)$

(i) $y \sim \chi^2(n)$

(j) $x_2 \sim \chi^2(\nu_2)$

(17) (a) By Problem 15, $v$ is multivariate normal

$$E(v) = E\left(\Sigma^{-\frac{1}{2}}(y-\mu)\right) = \Sigma^{-1/2}(E(y)-\mu) = \Sigma^{-\frac{1}{2}}(\mu-\mu) = 0$$

$$\mathrm{cov}(v) = \mathrm{cov}\left(\Sigma^{-\frac{1}{2}}(y-\mu)\right) = \mathrm{cov}\left(\Sigma^{-\frac{1}{2}}y\right)$$
$$= \Sigma^{-\frac{1}{2}}\mathrm{cov}(y)\,\Sigma^{-\frac{1}{2}\prime} = \Sigma^{-\frac{1}{2}}\,\Sigma\,\Sigma^{-\frac{1}{2}}$$
$$= \underbrace{\Sigma^{-\frac{1}{2}}\Sigma^{1/2}}\ \underbrace{\Sigma^{1/2}\Sigma^{-\frac{1}{2}}} = I_p$$

So $v \sim N_p(0, I_p)$

$$w = v'v = \left(\Sigma^{-\frac{1}{2}}(y-\mu)\right)'\Sigma^{-\frac{1}{2}}(y-\mu)$$
$$= (y-\mu)'\Sigma^{-\frac{1}{2}\prime}\Sigma^{-\frac{1}{2}}(y-\mu)$$
$$= (y-\mu)'\Sigma^{-\frac{1}{2}}\Sigma^{-\frac{1}{2}}(y-\mu)$$
$$= (y-\mu)'\Sigma^{-1}(y-\mu)$$

Since $v$ is a vector of independent standard normals,

$$v'v = \sum_{i=1}^{n} v_i^2 \sim \chi^2(p) \text{ by 16 h and g.}$$

(18) (a) Each $y_j$ is a linear combination of the $x_i$ values. The weights, whatever they are, are the rows of the $A$ matrix in $y = Ax$.

(b)
$$\operatorname{cov}(\bar{x}, x_j - \bar{x}) = \operatorname{cov}(\bar{x}, x_j) - \operatorname{cov}(\bar{x}, \bar{x})$$
$$= \operatorname{cov}(x_j, \bar{x}) - \operatorname{var}(\bar{x})$$
$$= \operatorname{cov}\left(x_j, \frac{1}{n}\sum_{i=1}^{n} x_i\right) - \frac{\sigma^2}{n}$$
$$= \frac{1}{n}\sum_{i=1}^{n} \operatorname{cov}(x_i, x_j) - \frac{\sigma^2}{n}$$
$$= \frac{1}{n}\left(\operatorname{cov}(x_j, x_j) + \sum_{i \neq j} \underbrace{\operatorname{cov}(x_i, x_j)}_{\text{Ind}}\right) - \frac{\sigma^2}{n}$$
$$= \frac{1}{n}\left(\operatorname{var}(x_j) + 0\right) - \frac{\sigma^2}{n} = \frac{\sigma^2}{n} - \frac{\sigma^2}{n} = 0$$

(c)
$$\operatorname{cov}(\bar{x}, x - j\bar{x}) = \operatorname{cov}\left(\frac{1}{n} j' x, x - j \frac{1}{n} j' x\right)$$
$$= \operatorname{cov}\left(\frac{1}{n} j' x, \left(I_n - \frac{1}{n} j j'\right) x\right)$$
(using $\operatorname{cov}(Aw, Bw) = A \operatorname{cov}(w) B'$ from Q16 A3)
$$= \frac{1}{n} j' \operatorname{cov}(x) \left(I_n - \frac{1}{n} j j'\right)'$$
$$= \frac{1}{n} j' \sigma^2 I_n \left(I_n - \frac{1}{n} j j'\right) = \sigma^2\left(\frac{1}{n} j' - \frac{1}{n^2} \underbrace{j' j}_{n} j'\right)$$
$$= \sigma^2\left(\frac{1}{n} j' - \frac{1}{n} j'\right) = \underset{\sim}{0}_{1 \times n}$$

(18 d) Because for the multivariate normal, zero covariance implies independence.

(e) Because $s^2$ is a function of $y_2$, and functions of independent random vectors are independent.

(f) $$\sum_{i=1}^{n}(x_i-\mu)^2 = \sum_{i=1}^{n}(x_i-\bar{x}+\bar{x}-\mu)^2$$

$$= \sum_{i=1}^{n}(x_i-\bar{x}) + 2(\bar{x}-\mu)\underbrace{\sum_{i=1}^{n}(x_i-\bar{x})}_{=0} + n(\bar{x}-\mu)^2$$

$$\Rightarrow \frac{1}{\sigma^2}\sum_{i=1}^{n}(x_i-\mu)^2 = \frac{\sum_{i=1}^{n}(x_i-\bar{x})^2}{\sigma^2} + \frac{n(\bar{x}-\mu)^2}{\sigma^2}$$

$$\sum_{i=1}^{n}\left(\frac{x_i-\mu}{\sigma}\right)^2 = \frac{(n-1)s^2}{\sigma^2} + \left(\frac{\bar{x}-\mu}{\sigma/\sqrt{n}}\right)^2$$

$\chi^2(n)$ $\chi^2(1)$

$$y = w_1 + w_2$$ From Q16 j

$w_1$ & $w_2$ are independent because they are functions of $s^2$ & $\bar{x}$, which are independent. So the conclusion follows by Q16 part j.

18 g Let $z = \frac{\sqrt{n}(\bar{x}-\mu)}{\sigma} \sim N(0,1)$

$$w = \frac{(n-1)s^2}{\sigma^2} \sim \chi^2(n-1)$$

Independent because $\bar{x}$ & $s^2$ are independent.
(This is where we use it.) So

$$t = \frac{z}{\sqrt{w/r}} = \frac{\frac{\sqrt{n}(\bar{x}-\mu)}{\cancel{\sigma}}}{\sqrt{\frac{\cancel{(n-1)}s^2}{\cancel{\sigma^2}} \Big/ \cancel{(n-1)}}}$$

$$= \frac{\sqrt{n}(\bar{x}-\mu)}{s} \sim t(n-1)$$

∎