STA 302f20 Assignment Six[1]

The following problems are not to be handed in. They are preparation for the Quiz in tutorial and the final exam. Please try them before looking at the answers. Use the formula sheet.

1. True or False: The sum of residuals is always equal to zero. Either prove that the statement is true, or use R to produce an example showing it is not true in general.

2. True or False: The sum of *expected* residuals is always equal to zero. Either prove that the statement is true, or use R to produce an example showing it is not true in general.

3. True or False: The sum of residuals is always equal to zero if the model has an intercept. Either prove that the statement is true, or use R to produce an example showing it is not true in general.

4. Sometimes one can learn by just playing around. Suppose we fit a regression model, obtaining $\widehat{\boldsymbol{\beta}}$, $\widehat{\mathbf{y}}$, $\widehat{\boldsymbol{\epsilon}}$ and so on. Then we fit another regression model with the same predictor variables, but this time using $\widehat{\mathbf{y}}$ as the predicted variable instead of $\mathbf{y}$.

   (a) Denote the vector of estimated regression coefficients from the new model by $\widehat{\widehat{\boldsymbol{\beta}}}$. Calculate $\widehat{\widehat{\boldsymbol{\beta}}}$ and simplify. Should you be surprised at this answer?

   (b) Calculate $\widehat{\widehat{\mathbf{y}}}$. Why is this not surprising if you think in terms of projections?

5. Now do the same thing as in the preceding question, but with $\widehat{\boldsymbol{\epsilon}}$ as the predicted variable. Can you understand this in terms of projections?

6. The joint moment-generating function of a p-dimensional random vector $\mathbf{x}$ is defined as $M_{\mathbf{x}}(\mathbf{t}) = E\left(e^{\mathbf{t}'\mathbf{x}}\right)$.

   (a) Let $\mathbf{y} = \mathbf{A}\mathbf{x}$, where $\mathbf{A}$ is a matrix of constants. Find the moment-generating function of $\mathbf{y}$.

   (b) Let $\mathbf{y} = \mathbf{x}+\mathbf{c}$, where $\mathbf{c}$ is a $p \times 1$ vector of constants. Find the moment-generating function of $\mathbf{y}$.

7. Let the random vector $\mathbf{x} = \left(\begin{array}{c}\mathbf{x}_1 \\ \hline \mathbf{x}_2\end{array}\right)$ and the vector of constants $\mathbf{t} = \left(\begin{array}{c}\mathbf{t}_1 \\ \hline \mathbf{t}_2\end{array}\right)$, with $\mathbf{t}_1$ the same length as $\mathbf{x}_1$, and $\mathbf{t}_2$ the same length as $\mathbf{x}_2$. Let $\mathbf{x}_1$ and $\mathbf{x}_2$ be independent, and for convenience also assume they have joint densities. Show that the joint moment-generating function $M_{\mathbf{x}}(\mathbf{t}) = M_{\mathbf{x}_1}(\mathbf{t}_1)M_{\mathbf{x}_2}(\mathbf{t}_2)$.

[1] This assignment was prepared by Jerry Brunner, Department of Statistical Sciences, University of Toronto. The LaTeX source code is available from the course website: http://www.utstat.toronto.edu/~brunner/oldclass/302f20

8. Let w_1 and w_2 be independent *scalar* random variables. Using moment-generating functions, show that $y_1 = g_1(w_1)$ and $y_2 = g_2(w_2)$ are independent. For convenience, you may assume that all the random variables have densities.

9. Let y be a *degenerate* random variable, with $P(y = \mu) = 1$.
   (a) Find the moment-generating function of y.
   (b) In what sense is y normally distributed?

10. Let $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \sim N_2(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, with $\boldsymbol{\mu} = \begin{pmatrix} 1 \\ 5 \end{pmatrix}$ and $\boldsymbol{\Sigma} = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$. Give non-zero constants a_1 and a_2 such that $y = a_1 x_1 + a_2 x_2$ has a degenerate distribution. Use moment-generating functions to show that the distribution is degenerate. This makes the joint distribution of $\mathbf{x}$ a *singular* multivariate normal.

11. Let $z_1, \ldots, z_p \stackrel{i.i.d.}{\sim} N(0, 1)$, and
$$\mathbf{z} = \begin{pmatrix} z_1 \\ \vdots \\ z_p \end{pmatrix}.$$
   (a) What is $E(\mathbf{z})$?
   (b) What is $cov(\mathbf{z})$?
   (c) What is the joint moment-generating function of $\mathbf{z}$? Show your work.
   (d) Let $\mathbf{y} = \boldsymbol{\Sigma}^{1/2}\mathbf{z} + \boldsymbol{\mu}$, where $\boldsymbol{\Sigma}$ is a $p \times p$ symmetric *non-negative definite* matrix and $\boldsymbol{\mu} \in \mathbb{R}^p$.
      i. What is $E(\mathbf{y})$?
      ii. What is the variance-covariance matrix of $\mathbf{y}$? Show some work.
      iii. What is the moment-generating function of $\mathbf{y}$? Show your work.

12. Let $\mathbf{y} \sim N_2(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, with
$$\mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \qquad \boldsymbol{\mu} = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} \qquad \boldsymbol{\Sigma} = \begin{pmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{pmatrix}$$
Using moment-generating functions, show y_1 and y_2 are independent. This is very similar to how the calculation goes for the full multivariate case.

13. Let $x_1 \sim N(1, 1)$, $x_2 \sim N(0, 2)$ and $x_3 \sim N(6, 1)$ be independent random variables, and let $y_1 = x_1 + x_2$ and $y_2 = x_2 + x_3$. Find the joint distribution of y_1 and y_2.

14. Let x_1 be Normal(μ_1, σ_1^2), and x_2 be Normal(μ_2, σ_2^2), independent of x_1. What is the joint distribution of $y_1 = x_1 + x_2$ and $y_2 = x_1 - x_2$? What is required for y_1 and y_2 to be independent?

15. Let $\mathbf{y} \sim N_p(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ and $\mathbf{w} = \mathbf{A}\mathbf{y} + \mathbf{c}$, where $\mathbf{A}$ is an $r \times p$ matrix of constants and $\mathbf{c}$ is an $r \times 1$ vector of constants. What is the distribution of $\mathbf{w}$? Prove your answer using moment-generating functions.

    You have shown that any linear transformation of a multivariate normal is multivariate normal. This is useful because from now on, if you observe that some random vector is a linear transformation of a multivariate normal, you don't need moment-generating functions to find its distribution. Just calculate the expected value and covariance matrix.

16. Here are some distribution facts that you will need to know without looking at a formula sheet in order to follow the proofs. You are responsible for the proofs of these facts too, but here you are just supposed to write down the answers.

    (a) Let $x \sim N(\mu, \sigma^2)$ and $y = ax + b$, where a and b are constants. What is the distribution of y?

    (b) Let $x \sim N(\mu, \sigma^2)$ and $z = \frac{x-\mu}{\sigma}$. What is the distribution of z?

    (c) Let $x_1, \ldots, x_n$ be a random sample from a $N(\mu, \sigma^2)$ distribution. What is the distribution of $y = \sum_{i=1}^n x_i$?

    (d) Let $x_1, \ldots, x_n$ be a random sample from a $N(\mu, \sigma^2)$ distribution. What is the distribution of the sample mean $\overline{x}$?

    (e) Let $x_1, \ldots, x_n$ be a random sample from a $N(\mu, \sigma^2)$ distribution. What is the distribution of $z = \frac{\sqrt{n}(\overline{x}-\mu)}{\sigma}$?

    (f) Let $x_1, \ldots, x_n$ be independent random variables, with $x_i \sim N(\mu_i, \sigma_i^2)$. Let $a_0, \ldots, a_n$ be constants. What is the distribution of $y = a_0 + \sum_{i=1}^n a_i x_i$?

    (g) Let $x_1, \ldots, x_n$ be independent random variables with $x_i \sim \chi^2(\nu_i)$ for $i = 1, \ldots, n$. What is the distribution of $y = \sum_{i=1}^n x_i$?

    (h) Let $z \sim N(0, 1)$. What is the distribution of $y = z^2$?

    (i) Let $x_1, \ldots, x_n$ be random sample from a $N(\mu, \sigma^2)$ distribution. What is the distribution of $y = \frac{1}{\sigma^2} \sum_{i=1}^n (x_i - \mu)^2$?

    (j) Let $y = w_1 + w_2$, where w_1 and w_2 are independent, $w_1 \sim \chi^2(\nu_1)$ and $y \sim \chi^2(\nu_1 + \nu_2)$. The parameters ν_1 and ν_2 are both positive. What is the distribution of w_2?

17. Let $\mathbf{y} \sim N_p(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, and $\mathbf{v} = \boldsymbol{\Sigma}^{-\frac{1}{2}}(\mathbf{y} - \boldsymbol{\mu})$.

    (a) As an application of Problem 15, what is the distribution of $\mathbf{v}$?

    (b) Show $w = (\mathbf{y} - \boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\mathbf{y} - \boldsymbol{\mu}) \sim \chi^2(p)$. This may be a bit easier than the way it was done in lecture.

18. Let $x_1, \ldots, x_n \stackrel{i.i.d}{\sim} N(\mu, \sigma^2)$, and $\mathbf{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$.

(a) Let $\mathbf{y} = \begin{pmatrix} x_1 - \overline{x} \\ \vdots \\ x_n - \overline{x} \\ \\ \overline{x} \end{pmatrix}$. How do you know that $\mathbf{y}$ is multivariate normal, without doing any calculations?

(b) Show $Cov\,(\overline{x}, x_j - \overline{x}) = 0$ This is a scalar calculation.

(c) Here is a matrix version of Question 18b. Following the text, let $\mathbf{j}$ denote a column vector of ones; in this case, $\mathbf{j}$ is $n \times 1$, and we can write $\overline{x} = \frac{1}{n}\mathbf{j}'\mathbf{x}$. The task is to show $cov\,(\overline{x}\,,\, \mathbf{x} - \mathbf{j}\overline{x}) = \mathbf{0}$. A hint is to use Question 16 of Assignment 3.

(d) Why does the preceding result (or Question 18b) show that $\overline{x}$ is independent of $\mathbf{y}_2 = \begin{pmatrix} x_1 - \overline{x} \\ \vdots \\ x_n - \overline{x} \end{pmatrix}$?

(e) Why does the independence of $\overline{x}$ and $\mathbf{y}_2$ imply the independence of $\overline{x}$ and s^2?

(f) Show that $\frac{(n-1)s^2}{\sigma^2} \sim \chi^2(n-1)$. Hint: $\sum_{i=1}^n (x_i - \mu)^2 = \sum_{i=1}^n (x_i - \overline{x} + \overline{x} - \mu)^2 = \ldots$. Where do you use the independence of $\overline{x}$ and s^2?

(g) Recall the definition of the t distribution. If $z \sim N(0,1)$, $w \sim \chi^2(\nu)$ and z and w are independent, then $t = \frac{z}{\sqrt{w/\nu}}$ is said to have a t distribution with ν degrees of freedom, and we write $t \sim t(\nu)$. For random sampling from a normal distribution, show that $t = \frac{\sqrt{n}(\overline{x}-\mu)}{s} \sim t(n-1)$. Where do you use the independence of $\overline{x}$ and s^2?