Assignment 3

(1) $tr(AB) = tr\left(\sum_{k=1}^{q} a_{ik} b_{kj}\right) = \sum_{i=1}^{p} \sum_{k=1}^{q} a_{ik} b_{ki}$

$= \sum_{k=1}^{q} \sum_{i=1}^{p} b_{ki} a_{ik} = tr(BA)$

switch both

There was more detail in the lecture slides. This is enough.

(2) If $AB = I$, both A & B must have inverses, for otherwise,

$|AB| = |A||B| = 0 \neq 1 = |I|$

Then $AB = I \Rightarrow A^{-1}AB = A^{-1}I \Rightarrow B = A^{-1}$ and

$AB = I \Rightarrow ABB^{-1} = IB^{-1} \Rightarrow A = B^{-1}$ ☒

(3)

(2.28) Thm 2.5a says $(A')^{-1} = (A^{-1})'$

Set $B = A^{-1}$. Need to show $B'A' = I$.

$B'A' = (AB)' = I' = I$ done.

(2.35) Need to show $y'Ay = y'A'y$.

$y'A'y$ is 1×1 & hence symmetric, so

$y'A'y = (y'A'y)' = y'A''y'' = y'Ay$ ☒

(2.36) $v'(P'AP)v = \underbrace{(v'P')}_{z'} A \underbrace{Pv}_{z} = z'Az \geq 0$

Because A is positive semi-definite

(4) (2.38) Because A is symmetric, $A = CDC'$. Since A is positive definite, the eigenvalues in D are positive and $A = CD^{1/2}D^{1/2}C' = CD^{1/2}(CD^{1/2})'$ $= P'P$, where $P = D^{1/2}C'$. P is non-singular, because $P^{-1} = CD^{-1/2}$. The inverse $D^{-1/2}$ exists because the eigenvalues are strictly positive.

(5) (2.53) Thm 2.9c says if A & B are square & the same size, $|AB| = |A|\,|B|$

<u>Cor 1</u> $|AB| = |A|\,|B| = |B|\,|A| = |BA|$

<u>Cor 2</u> $|A^2| = |AA| = |A|\,|A| = |A|^2$

(2.76) (2.109) says if A is pos definite, $A^{1/2} = CD^{1/2}C'$

(a) $(A^{1/2})' = (CD^{1/2}C')' = C''(D^{1/2})'C' = CD^{1/2}C'$
$= A^{1/2}$

(b) $A^{\frac{1}{2}}A^{1/2} = CD^{1/2}\underbrace{C'C}_{I}D^{1/2}C'$

$= CD^{1/2}D^{1/2}C' = CDC' = A$

6.

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> A = rbind(c(1.00, 0.75, 0.75, 0.75),
+           c(0.75, 1.00, 0.75, 0.75),
+           c(0.75, 0.75, 1.00, 0.75),
+           c(0.75, 0.75, 0.75, 1.00) ); A
     [,1] [,2] [,3] [,4]
[1,] 1.00 0.75 0.75 0.75
[2,] 0.75 1.00 0.75 0.75
[3,] 0.75 0.75 1.00 0.75
[4,] 0.75 0.75 0.75 1.00
>
> # a)
> c(det(solve(A)), 1/det(A))
[1] 19.69231 19.69231
>
> # b)
> c(det(A%*%A), det(A)^2)
[1] 0.002578735 0.002578735
>
> # c)
> eigen(A)
$values
[1] 3.25 0.25 0.25 0.25

$vectors
     [,1]       [,2]       [,3]        [,4]
[1,] -0.5  0.0000000  0.5855001 -0.63811414
[2,] -0.5  0.0000000  0.4064531  0.76471949
[3,] -0.5 -0.7071068 -0.4959766 -0.06330268
[4,] -0.5  0.7071068 -0.4959766 -0.06330268

>
> # d) A^{1/2}
> C = eigen(A)$vectors; D = diag(eigen(A)$values)
> Ahalf = C %*% sqrt(D) %*% t(C); Ahalf
          [,1]      [,2]      [,3]      [,4]
[1,] 0.8256939 0.3256939 0.3256939 0.3256939
[2,] 0.3256939 0.8256939 0.3256939 0.3256939
[3,] 0.3256939 0.3256939 0.8256939 0.3256939
[4,] 0.3256939 0.3256939 0.3256939 0.8256939
> Ahalf %*% Ahalf # Should be A
     [,1] [,2] [,3] [,4]
[1,] 1.00 0.75 0.75 0.75
[2,] 0.75 1.00 0.75 0.75
[3,] 0.75 0.75 1.00 0.75
[4,] 0.75 0.75 0.75 1.00
>
> # e) A^{-1/2}
> Aminushalf = C %*% solve(sqrt(D)) %*% t(C); Aminushalf
          [,1]      [,2]      [,3]      [,4]
[1,]  1.638675 -0.361325 -0.361325 -0.361325
[2,] -0.361325  1.638675 -0.361325 -0.361325
[3,] -0.361325 -0.361325  1.638675 -0.361325
[4,] -0.361325 -0.361325 -0.361325  1.638675
> Aminushalf %*% Ahalf # Should be I
              [,1]          [,2]          [,3]          [,4]
[1,]  1.000000e+00 -3.747003e-16 -5.551115e-17 -1.110223e-16
[2,] -1.387779e-16  1.000000e+00  9.575674e-16  4.440892e-16
[3,]  3.469447e-16  1.110223e-15  1.000000e+00  1.665335e-16
[4,]  1.110223e-16  5.551115e-16  1.110223e-16  1.000000e+00
```

(7) If cols of A are linearly dependent, there is $v \neq 0$ with $Av = 0$. If A^{-1} exists
$A^{-1}Av = A^{-1}0 \Rightarrow v = 0$, contradicting $v \neq 0$

(8) $a'a = \sum_{i=1}^{n} a_i^2 \geq 0$

(9) (a) (i) $D^{-1} = \begin{pmatrix} 1/\lambda_1 & 0 & \cdots & 0 \\ 0 & 1/\lambda_2 & 0 & \cdots \\ & & \ddots & \\ & & & 1/\lambda_k \end{pmatrix}$

(ii) $C D^{-1} \underbrace{C' C}_{I} D C' = C D^{-1} D C' = C I C' = C C' = I$

(b) (i) $D^{1/2} = \begin{pmatrix} \sqrt{\lambda_1} & & & 0 \\ & \sqrt{\lambda_2} & & \\ & & \ddots & \\ 0 & & & \sqrt{\lambda_k} \end{pmatrix}$

(ii) $(\Sigma^{1/2})' = (C D^{1/2} C')' = C'' (D^{1/2})' C'$

$= C D^{1/2} C'$

iii) $\Sigma^{1/2} \Sigma^{1/2} = C D^{1/2} \underbrace{C' C}_{I} D^{1/2} C'$

$= C D^{1/2} D^{1/2} C'$

$= C D C' = \Sigma$

(9c) (i) $\Sigma^{-1/2}\Sigma^{1/2} = CD^{-1/2}\underbrace{C'C}_{I}D^{1/2}C'$

$= C\underbrace{D^{-1/2}D^{1/2}}_{I}C' = CC' = I$

(ii) $\Sigma^{-\frac{1}{2}}\Sigma^{-\frac{1}{2}} = CD^{-1/2}\underbrace{C'C}_{I}D^{-1/2}C'$

$= CD^{-1/2}D^{-1/2}C' = CD^{-1}C' = \Sigma^{-1}$

(d) λ an eigenvalue of Σ means $Ax = \lambda x$, where x is the corresponding eigenvector.

$Ax = \lambda x \Rightarrow x'Ax = x'\lambda x = \lambda x'x = \lambda \cdot 1$

$\Rightarrow \lambda = x'Ax > 0$

(e) $\Sigma = CDC'$ by spectral decomposition.

$CD^{-1}\underbrace{C'C}_{I}DC' = C\underbrace{D^{-1}D}_{I}C' = CC' = I$

So $\Sigma^{-1} = CD^{-1}C'$ exists

(10) Let Σ be symmetric

$$tr(\Sigma) = tr(\underbrace{CD}_{A}\underbrace{C'}_{B}) = tr(\underbrace{C'C}_{I}D)$$

$= tr(D) =$ sum of eigenvalues, since D is a diagonal matrix of eigenvalues.

(11) $$|\Sigma| = |\underbrace{C}_{A}\underbrace{DC'}_{B}| = |D\underbrace{C'C}_{I}| = |D|$$

Product of eigenvalues

(12) If the column vector v has a one in position j and all the rest zeros, then $v'A$ is row j of A, and $v'Av$ is $a_{jj} > 0$ because A is positive definite.

(13) $E(XB) = E\left(\left[\sum_k x_{ik} b_{kj}\right]\right)$

$= \left[E\left(\sum_k x_{ik} b_{kj}\right)\right] = \left[\sum_k E(x_{ik}) b_{kj}\right]$

$= E(X)B$

(14) $\operatorname{cov}(y) \overset{def}{=} E\left((y-\mu)(y-\mu)'\right) = E\left((y-\mu)(y'-\mu')\right)$

$= E\left(yy' - y\mu' - \mu y' + \mu\mu'\right)$

$= E(yy') - E(y)\mu' - \mu E(y') + \mu\mu'$

$= E(yy') - \cancel{\mu\mu'} - \mu E(y)' + \cancel{\mu\mu'}$

$= E(yy') - \mu\mu'$

(15) $\operatorname{cov}(Ay) = E\left((Ay - A\mu)(Ay - A\mu)'\right)$

$= E\left(A(y-\mu)\left[A(y-\mu)\right]'\right)$

$= E\left(A(y-\mu)(y-\mu)'A'\right)$

$= A\,E\left((y-\mu)(y-\mu)'\right)A' = A\Sigma A'$

The two expressions are <u>not</u> equal!

16. $\operatorname{cov}(Ay, By) = E\left((Ay - A\mu)(By - B\mu)'\right)$

$= E\left(A(y-\mu)(B(y-\mu))'\right) = E\left(A(y-\mu)(y-\mu)'B'\right)$

$= A\,E\left((y-\mu)(y-\mu)'\right)B' = A\,\Sigma B'$

17. Just kidding, but some people can prove anything.

18. Noting $E(Ax + c) = AE(x) + c = A\mu + c$,

$\operatorname{cov}(Ax + c) = E\left((Ax + \cancel{c} - A\mu - \cancel{c})(Ax + \cancel{c} - A\mu - \cancel{c})'\right)$

$= E\left(A(x-\mu)(A(x-\mu))'\right) = E\left(A(x-\mu)(x-\mu)'A'\right)$

$= A\,E\left((x-\mu)(x-\mu)'\right)A' = A\,\Sigma A'$

19. $\operatorname{Var}(Y) = \operatorname{cov}(v'x) = v'\operatorname{cov}(x)\,v = v'\Sigma v \geq 0$ because any variance ≥ 0, so Σ is positive semi-definite. It could be the covariance matrix of <u>any</u> random vector.

20. (a) $\Sigma v = \lambda v \Rightarrow v'\Sigma v = v'\lambda v = \lambda v'v$

$\Longrightarrow \lambda = v'\Sigma v / v'v$ since $v \neq 0$

≥ 0 because by problem 19, Σ is positive semi definite. ▫

(20 b) Because the determinant is the product of eigenvalues, and by part (a) all the eigenvalues are non-negative.

(c) $\operatorname{cov}\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} \operatorname{Var}(X) & \operatorname{Cov}(X,Y) \\ \operatorname{cov}(X,Y) & \operatorname{Var}(Y) \end{pmatrix} = \Sigma$

By part (b), $0 \le |\Sigma| = \operatorname{Var}(X)\operatorname{Var}(Y) - \operatorname{cov}(X,Y)^2$

$\Longrightarrow \operatorname{Var}(X)\operatorname{Var}(Y) \ge \operatorname{Cov}(X,Y)^2$, which is actually the Cauchy-Schwarz inequality

$$\Longrightarrow \frac{\operatorname{Cov}(X,Y)^2}{\operatorname{Var}(X)\operatorname{Var}(Y)} \le 1 \Longrightarrow \left| \frac{\operatorname{cov}(X,Y)}{\sqrt{\operatorname{Var}(X)\operatorname{Var}(Y)}} \right| \le 1$$

$$\Longrightarrow |\operatorname{Corr}(X,Y)| \le 1 \Longleftrightarrow -1 \le \operatorname{Corr}(X,Y) \le 1$$

(21) (a) $[\mathrm{cov}(x,y)] = \mathrm{Cov}(X_i, Y_j)$

(b) $$\mathrm{cov}(x+y) = E\{(x+y-\mu_x-\mu_y)(x+y-\mu_x-\mu_y)'\}$$

$$= E\{(x-\mu_x + y-\mu_y)(x-\mu_x + y-\mu_y)'\}$$

$$= E\{(x-\mu_x)(x-\mu_x)' + (x-\mu_x)(y-\mu_y)' + (y-\mu_y)(x-\mu_x)' + (y-\mu_y)(y-\mu_y)'\}$$

$$= E\{(x-\mu_x)(x-\mu_x)'\} + E\{(x-\mu_x)(y-\mu_y)'\} + E\{(y-\mu_y)(x-\mu_x)'\} + E\{(y-\mu_y)(y-\mu_y)'\}$$

$$= \mathrm{cov}(x) + \mathrm{cov}(x,y) + \mathrm{cov}(y,x) + \mathrm{cov}(y)$$

$$= \Sigma_x + \Sigma_{xy} + \Sigma_{xy}' + \Sigma_y$$

(c) If $\mathrm{Cov}(X_i, Y_j)$ for all $i \neq j$, $\mathrm{Cov}(x+y) = \Sigma_x + \Sigma_y$

(d) $$\mathrm{Cov}(x+c, y+d) = E\{(x+\cancel{c}-\mu_x-\cancel{c})(y+\cancel{d}-\mu_y-\cancel{d})'\}$$

$$= E\{(x-\mu_x)(y-\mu_y)'\} = \mathrm{cov}(x,y)$$

(22) $\mathrm{cov}(x_1+x_2, y_1+y_2) = E\{(x_1+x_2-\mu_1-\mu_2)(y_1+y_2-\mu_3-\mu_4)'\}$

$= E\{(x_1-\mu_1 + x_2-\mu_2)(y_1-\mu_3 + y_2-\mu_4)'\}$

$= E\{(x_1-\mu_1)(y_1-\mu_3)'\} + E\{(x_1-\mu_1)(y_2-\mu_4)'\}$

$+ E\{(x_2-\mu_2)(y_1-\mu_3)'\} + E\{(x_2-\mu_2)(y_2-\mu_4)'\}$

$= \mathrm{cov}(x_1, y_1) + \mathrm{cov}(x_1, y_2) + \mathrm{cov}(x_2, y_1) + \mathrm{cov}(x_2, y_2)$

(23) It's not true, even if $p=q$. There is no reason to think $\mathrm{Cov}(X_i, Y_j) = \mathrm{Cov}(Y_i, X_j)$

(24) This was supposed to be a joke.

(25) (3.20) (a) Let $v = (2,-3,1)'$. Then $z = v'y$, and

$E(z) = E(v'y) = v'E(y) = (2,-3,1)\begin{pmatrix}1\\-1\\3\end{pmatrix} = 8$

$\mathrm{var}(z) = \mathrm{cov}(v'y) = v'\Sigma v$

$= (2,-3,1)\begin{pmatrix}1 & 1 & 0\\ 1 & 2 & 3\\ 0 & 3 & 10\end{pmatrix}\begin{pmatrix}2\\-3\\1\end{pmatrix} = (-1,-1,1)\begin{pmatrix}2\\-3\\1\end{pmatrix} =$

$= -2+3+1 = 2$

(Q 25, 3.20 b)

$$z = Ay, \quad A = \begin{pmatrix} 1 & 1 & 1 \\ 3 & 1 & -2 \end{pmatrix}$$

$$E(z) = A E(y) = A\mu = \begin{pmatrix} 1 & 1 & 1 \\ 3 & 1 & -2 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix}$$

$$= \begin{pmatrix} 3 \\ 3-1-6 \end{pmatrix} = \begin{pmatrix} 3 \\ -4 \end{pmatrix}$$

$$\operatorname{cov}(z) = \operatorname{cov}(Ay) = A \Sigma A'$$

$$= \begin{pmatrix} 1 & 1 & 1 \\ 3 & 1 & -2 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 3 \\ 0 & 3 & 10 \end{pmatrix} \begin{pmatrix} 1 & 3 \\ 1 & 1 \\ 1 & -2 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 6 & 13 \\ 4 & -1 & -17 \end{pmatrix} \begin{pmatrix} 1 & 3 \\ 1 & 1 \\ 1 & -2 \end{pmatrix} = \begin{pmatrix} 21 & -14 \\ -14 & 45 \end{pmatrix}$$