STA 302f20 Assignment Three[1]

Please do these review questions in preparation for Quiz Three; they are not to be handed in. Use the current formula sheet on the course website. Questions 1-12 are based on material in Chapter 2. Questions 13-25 are based on material in Chapter 3.

1. Recall that the *trace* of a square matrix is the sum of diagonal elements. So if $\mathbf{C} = (c_{ij})$ is a $p \times p$ matrix, $tr(\mathbf{C}) = \sum_{j=1}^{p} c_{jj}$. Let $\mathbf{A}$ be a $p \times q$ constant matrix, and let $\mathbf{B}$ be a $q \times p$ constant matrix, so that $\mathbf{AB}$ and $\mathbf{BA}$ are both defined. Prove $tr(\mathbf{AB}) = tr(\mathbf{BA})$.

2. Let $\mathbf{A}$ and $\mathbf{B}$ be square matrices of constants, with $\mathbf{AB} = \mathbf{I}$. Using $|\mathbf{AB}| = |\mathbf{A}|\,|\mathbf{B}|$, prove $\mathbf{BA} = \mathbf{I}$. Thus when you are showing that a matrix is the inverse of another matrix, you only need to multiply them in one direction and get the identity.

3. In the textbook, do Problems 2.28, 2.35 and 2.36.

4. In the textbook, do Problem 2.38. It is asking you to show that if the symmetric matrix $\mathbf{A}$ is positive definite, then $\mathbf{A} = \mathbf{P}'\mathbf{P}$ for some non-singular $\mathbf{P}$.

5. In the textbook, do Problems 2.53 and 2.76.

6. Let $\mathbf{A} = \left(\begin{array}{cccc} 1.00 & 0.75 & 0.75 & 0.75 \\ 0.75 & 1.00 & 0.75 & 0.75 \\ 0.75 & 0.75 & 1.00 & 0.75 \\ 0.75 & 0.75 & 0.75 & 1.00 \end{array} \right)$. Enter $\mathbf{A}$ into R using `rbind`.

   (a) Calculate $|\mathbf{A}^{-1}|$ and $1/|\mathbf{A}|$, verifying that they are equal.

   (b) Calculate $|\mathbf{A}^2|$ and $|\mathbf{A}|^2$, verifying that they are equal.

   (c) Calculate the eigenvalues and eigenvectors of $\mathbf{A}$.

   (d) Calculate $\mathbf{A}^{1/2}$. Multiply the matrix by itself to get $\mathbf{A}$.

   (e) Calculate $\mathbf{A}^{-1/2}$. Multiply the matrix by $\mathbf{A}^{1/2}$ to get the identity.

   Display the creation of $\mathbf{A}$, and also the input and output for each part. Label the output with comments. Be ready to hand in a pdf with the quiz if requested.

7. Let $\mathbf{A}$ be a square matrix. Show that if the columns of $\mathbf{A}$ are linearly dependent, $\mathbf{A}^{-1}$ cannot exist. Hint: $\mathbf{v}$ cannot be both zero and not zero at the same time.

8. Let $\mathbf{a}$ be an $n \times 1$ matrix of real constants. How do you know $\mathbf{a}'\mathbf{a} \geq 0$?

[1]This assignment was prepared by Jerry Brunner, Department of Statistical Sciences, University of Toronto. The LaTeX source code is available from the course website: `http://www.utstat.toronto.edu/~brunner/oldclass/302f20`

9. Recall the *spectral decomposition* of a square symmetric matrix (For example, a variance-covariance matrix). Any such matrix $\mathbf{\Sigma}$ can be written as $\mathbf{\Sigma} = \mathbf{CDC}'$, where $\mathbf{C}$ is a matrix whose columns are the (orthonormal) eigenvectors of $\mathbf{\Sigma}$, $\mathbf{D}$ is a diagonal matrix of the corresponding eigenvalues, and $\mathbf{C}'\mathbf{C} = \mathbf{CC}' = \mathbf{I}$.

   (a) Let $\mathbf{\Sigma}$ be a square symmetric matrix with eigenvalues that are all strictly positive.
      i. What is $\mathbf{D}^{-1}$?
      ii. Show $\mathbf{\Sigma}^{-1} = \mathbf{CD}^{-1}\mathbf{C}'$

   (b) Let $\mathbf{\Sigma}$ be a square symmetric matrix, and this time some of the eigenvalues might be zero.
      i. What do you think $\mathbf{D}^{1/2}$ might be?
      ii. Define $\mathbf{\Sigma}^{1/2}$ as $\mathbf{CD}^{1/2}\mathbf{C}'$. Show $\mathbf{\Sigma}^{1/2}$ is symmetric.
      iii. Show $\mathbf{\Sigma}^{1/2}\mathbf{\Sigma}^{1/2} = \mathbf{\Sigma}$.

   (c) Now return to the situation where the eigenvalues of the square symmetric matrix $\mathbf{\Sigma}$ are all strictly positive. Define $\mathbf{\Sigma}^{-1/2}$ as $\mathbf{CD}^{-1/2}\mathbf{C}'$, where the elements of the diagonal matrix $\mathbf{D}^{-1/2}$ are the reciprocals of the corresponding elements of $\mathbf{D}^{1/2}$.
      i. Show that the inverse of $\mathbf{\Sigma}^{1/2}$ is $\mathbf{\Sigma}^{-1/2}$, justifying the notation.
      ii. Show $\mathbf{\Sigma}^{-1/2}\mathbf{\Sigma}^{-1/2} = \mathbf{\Sigma}^{-1}$.

   (d) The (square) matrix $\mathbf{\Sigma}$ is said to be *positive definite* if $\mathbf{v}'\mathbf{\Sigma}\mathbf{v} > 0$ for all vectors $\mathbf{v} \neq \mathbf{0}$. Show that the eigenvalues of a positive definite matrix are all strictly positive.

   (e) Let $\mathbf{\Sigma}$ be a symmetric, positive definite matrix. Putting together a couple of results you have proved above, establish that $\mathbf{\Sigma}^{-1}$ exists.

10. Using the Spectral Decomposition Theorem and $tr(\mathbf{AB}) = tr(\mathbf{BA})$, prove that the trace is the sum of the eigenvalues for a symmetric matrix.

11. Using the Spectral Decomposition Theorem and $|\mathbf{AB}| = |\mathbf{BA}|$, prove that the determinant of a symmetric matrix is the product of its eigenvalues.

12. Prove that the diagonal elements of a positive definite matrix must be positive. Hint: Can you describe a vector $\mathbf{v}$ such that $\mathbf{v}'\mathbf{Av}$ picks out the jth diagonal element?

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13. Let $\mathbf{X}$ be a random matrix, and $\mathbf{B}$ be a matrix of constants. Show $E(\mathbf{XB}) = E(\mathbf{X})\mathbf{B}$.

14. Do Problem 3.10 in the text.

15. Let the $p \times 1$ random vector $\mathbf{x}$ have expected value $\boldsymbol{\mu}$ and variance-covariance matrix $\mathbf{\Sigma}$, and let $\mathbf{A}$ be an $m \times p$ matrix of constants. Prove that the variance-covariance matrix of $\mathbf{Ax}$ is either

    - $\mathbf{A\Sigma A}'$, or
    - $\mathbf{A}^2\mathbf{\Sigma}$..

    Pick one and prove it. Start with the definition of a variance-covariance matrix on the formula sheet. If the two expressions are equal, say so.

16. Let the $p \times 1$ random vector $\mathbf{y}$ have expected value $\boldsymbol{\mu}$ and variance-covariance matrix $\boldsymbol{\Sigma}$. Find $cov(\mathbf{Ay}, \mathbf{By})$, where A and B are matrices of constants.

17. Let $\mathbf{x}$ be a $p \times 1$ random vector. Starting with the definition on the formula sheet, prove $cov(\mathbf{x}) = \mathbf{0}$..

18. Let the $p \times 1$ random vector $\mathbf{x}$ have mean $\boldsymbol{\mu}$ and variance-covariance matrix $\boldsymbol{\Sigma}$, let $\mathbf{A}$ be an $r \times p$ matrix of constants, and let $\mathbf{c}$ be an $r \times 1$ vector of constants. Find $cov(\mathbf{Ax} + \mathbf{c})$. Show your work.

19. Comparing the definitions, one can see that viewing a scalar random variable as a 1×1 random vector, the variance-covariance matrix is just the ordinary variance. Accordingly, let the scalar random variable $Y = \mathbf{v}'\mathbf{x}$, where $\mathbf{x}$ is a $p \times 1$ random vector with covariance matrix $\boldsymbol{\Sigma}$, and $\mathbf{v}$ is a $p \times 1$ vector of constants. What is $Var(Y)$? Why does this tell you that *any* variance-covariance matrix must be positive semi-definite?

20. Using definitions on the formula sheet and other material from this assignment,
    (a) Show that the eigenvalues of a variance-covariance matrix cannot be negative.
    (b) How do you know that the determinant of a variance-covariance matrix must be greater than or equal to zero? The answer is one short sentence.
    (c) Let X and Y be scalar random variables. Recall $Corr(X,Y) = \frac{Cov(X,Y)}{\sqrt{Var(X)Var(Y)}}$. Using what you have shown about the determinant, show $-1 \leq Corr(X,Y) \leq 1$. You have just proved the Cauchy-Schwarz inequality.

21. Let $\mathbf{x}$ be a $p \times 1$ random vector with mean $\boldsymbol{\mu}_x$ and variance-covariance matrix $\boldsymbol{\Sigma}_x$, and let $\mathbf{y}$ be a $q \times 1$ random vector with mean $\boldsymbol{\mu}_y$ and variance-covariance matrix $\boldsymbol{\Sigma}_y$.
    (a) What is the (i,j) element of $\boldsymbol{\Sigma}_{xy} = cov(\mathbf{x}, \mathbf{y})$?
    (b) Find an expression for $cov(\mathbf{x} + \mathbf{y})$ in terms of $\boldsymbol{\Sigma}_x$, $\boldsymbol{\Sigma}_y$ and $\boldsymbol{\Sigma}_{xy}$. Show your work.
    (c) Simplify further for the special case where $Cov(X_i, Y_j) = 0$ for all i and j.
    (d) Let $\mathbf{c}$ be a $p \times 1$ vector of constants and $\mathbf{d}$ be a $q \times 1$ vector of constants. Find $cov(\mathbf{x} + \mathbf{c}, \mathbf{y} + \mathbf{d})$. Show your work.

22. Let the random vectors $\mathbf{x}_1$ and $\mathbf{x}_2$ be $p \times 1$, and the random vectors $\mathbf{y}_1$ and $\mathbf{y}_2$ be $p \times 1$, with $E(\mathbf{x}_1) = \boldsymbol{\mu}_1$, $E(\mathbf{x}_2) = \boldsymbol{\mu}_2$, $E(\mathbf{y}_1) = \boldsymbol{\mu}_3$, $E(\mathbf{y}_2) = \boldsymbol{\mu}_4$. Show that the $p \times q$ matrix of covariances $cov(\mathbf{x}_1 + \mathbf{x}_2, \mathbf{y}_1 + \mathbf{y}_2) = cov(\mathbf{x}_1, \mathbf{y}_1) + cov(\mathbf{x}_1, \mathbf{y}_2) + cov(\mathbf{x}_2, \mathbf{y}_1) + cov(\mathbf{x}_2, \mathbf{y}_2)$.

23. Starting with the definition on the formula sheet, show $cov(\mathbf{x}, \mathbf{y}) = cov(\mathbf{y}, \mathbf{x})$..

24. Starting with the definition on the formula sheet, show $cov(\mathbf{x}, \mathbf{y}) = \mathbf{0}$..

25. Do problem 3.20 in the text. The answer is in the back of the book.