STA 302f20 Assignment Two[1]

Please do these review questions in preparation for Quiz Two; they are not to be handed in. Use the formula sheet on the course website. Starting with Problem 4, you can play a little game. Try not to do the same work twice. Instead, use results of earlier problems whenever possible.

1. Read Chapter 2 in *Linear models in statistics*, optionally skipping Sections 2.8 (generalized inverses), 2.13 (idempotent matrices) and 2.14 (vector and matrix calculus). Do problems 2.6a, 2.6d, 2.7b, 2.7c, 2.17c, 2.17d, 2.20, 2.23, 2.24. You should be able to do these problems without reading anything, but the assigned reading will help, soon.

2. Let $\mathbf{A}$ by a non-singular square matrix. Prove that $\mathbf{A}^{-1}$ is unique by letting both $\mathbf{B}$ and $\mathbf{C}$ be inverses of $\mathbf{A}$, and then showing $\mathbf{B} = \mathbf{C}$.

3. This problem is more review, this time of statistical concepts you encountered in STA260 and probably STA258. Let $y_1, \ldots, y_n$ be a random sample (that is, independent and identically distributed) from a normal distribution with mean μ and variance σ^2, so that $t = \frac{\sqrt{n}(\overline{y}-\mu)}{s} \sim t(n-1)$. This is something you don't need to prove, for now.

   (a) Derive a $(1-\alpha)100\%$ confidence interval for μ. "Derive" means show all the high school algebra. Use the symbol $t_{1-\alpha/2}$ for the number satisfying $Pr(T \leq t_{1-\alpha/2}) = 1 - \alpha/2$.

   (b) A random sample with $n = 23$ yields $\overline{y} = 2.57$ and a sample variance of $s^2 = 5.85$.

      i. Use R to find the critical value $t_{0.975}$.

      ii. Give a 95% confidence interval for μ. The answer is a pair of numbers, the lower confidence limit and the upper confidence limit.

   (c) Using the sample statistics from Question 3b, test $H_0 : \mu = 3$ versus $H_1 : \mu \neq 3$ at $\alpha = 0.05$.

      i. Give the value of the T statistic. The answer is a number.

      ii. What is the critical value? The answer is a number.

      iii. State whether you reject H_0, Yes or No.

      iv. What is the p-value? Give the number and the R command that produced it.

      v. Can you conclude that μ is different from 3? Answer Yes or No.

      vi. If the answer is Yes, state whether $\mu > 3$ or $\mu < 3$. Pick one.

[1]This assignment was prepared by Jerry Brunner, Department of Statistical Sciences, University of Toronto. The LaTeX source code is available from the course website: http://www.utstat.toronto.edu/~brunner/oldclass/302f20

4. Denote the moment-generating function of a random variable y by $M_y(t)$. The moment-generating function is defined by $M_y(t) = E(e^{yt})$.

   (a) Let a be a constant. Prove that $M_{ax}(t) = M_x(at)$.

   (b) Prove that $M_{x+a}(t) = e^{at}M_x(t)$.

   (c) Let $x_1, \ldots, x_n$ be *independent* random variables. Prove that

$$M_{\sum_{i=1}^n x_i}(t) = \prod_{i=1}^n M_{x_i}(t).$$

   Clearly indicate where you use independence.

5. Recall that if $x \sim N(\mu, \sigma^2)$, it has moment-generating function $M_x(t) = e^{\mu t + \frac{1}{2}\sigma^2 t^2}$. You will not have to prove this.

   (a) Let $x \sim N(\mu, \sigma^2)$ and $y = ax + b$, where a and b are constants. Use moment-generating functions to find the distribution of y. Show your work.

   (b) Let $x \sim N(\mu, \sigma^2)$ and $z = \frac{x-\mu}{\sigma}$. Use moment-generating functions to find the distribution of z. Show your work.

   (c) Let $x_1, \ldots, x_n$ be a random sample from a $N(\mu, \sigma^2)$ distribution. Use moment-generating functions to find the distribution of $y = \sum_{i=1}^n x_i$. Show your work.

   (d) Let $x_1, \ldots, x_n$ be a random sample from a $N(\mu, \sigma^2)$ distribution. Use moment-generating functions to find the distribution of the sample mean $\overline{x}$.

   (e) Let $x_1, \ldots, x_n$ be a random sample from a $N(\mu, \sigma^2)$ distribution. Find the distribution of $z = \frac{\sqrt{n}(\overline{x}-\mu)}{\sigma}$. Show your work.

   (f) Let $x_1, \ldots, x_n$ be independent random variables, with $x_i \sim N(\mu_i, \sigma_i^2)$. Let $a_0, \ldots, a_n$ be constants. Use moment-generating functions to find the distribution of $y = a_0 + \sum_{i=1}^n a_i x_i$. Show your work. This is a big deal, because it establishes that any linear combinations of independent normals is normal. Thus, to find the distribution of any linear combination of independent normals, all you need to do is calculate the expected value and variance.

6. A Chi-squared random variable x with parameter $\nu > 0$ has moment-generating function $M_x(t) = (1-2t)^{-\nu/2}$ for $t < 1/2$. You will not have to prove this.

   (a) Let $x_1, \ldots, x_n$ be independent random variables with $x_i \sim \chi^2(\nu_i)$ for $i = 1, \ldots, n$. Find the distribution of $y = \sum_{i=1}^n x_i$.

   (b) Let $z \sim N(0, 1)$. Find the distribution of $y = z^2$ using moment-generating functions. For this one, you need to integrate.

   (c) Let $x_1, \ldots, x_n$ be random sample from a $N(\mu, \sigma^2)$ distribution. Find the distribution of $y = \frac{1}{\sigma^2}\sum_{i=1}^n (x_i - \mu)^2$.

   (d) Let $y = x_1 + x_2$, where x_1 and x_2 are independent, $x_2 \sim \chi^2(\nu_2)$ and $y \sim \chi^2(\nu_1 + \nu_2)$, where ν_1 and ν_2 are both positive. Show $x_1 \sim \chi^2(\nu_1)$.

(e) Let $x_1, \ldots, x_n$ be random sample from a $N(\mu, \sigma^2)$ distribution. Show

$$\frac{(n-1)s^2}{\sigma^2} \sim \chi^2(n-1),$$

where $s^2 = \frac{\sum_{i=1}^n (x_i - \overline{x})^2}{n-1}$. Hint: $\sum_{i=1}^n (x_i - \mu)^2 = \sum_{i=1}^n (x_i - \overline{x} + \overline{x} - \mu)^2 = \ldots$
For this question, you may use the independence of $\overline{x}$ and s^2 without proof. We will prove it later. Note: This is a special case of a central result that will be used throughout most of the course.

7. We return to simple linear regression (see problem 14 from last week). "Simple" means that there is just one explanatory variable. Here's the model. Independently for $i = 1, \ldots, n$, let $y_i = \beta_0 + \beta_1 x_i + \epsilon_i$, where β_0 and β_1 are unknown constants (parameters), $x_1, \ldots, x_n$ are a known, observable constants, and $\epsilon_1, \ldots, \epsilon_n$ are independent random variables with expected value zero and unknown variance σ^2.

(a) In *least squares* estimation, one first writes the expected value of y_i as a function of $\boldsymbol{\beta} = (\beta_0, \beta_1)$, say $E_{\boldsymbol{\beta}}(y_i)$, and then one estimates the β_j by choosing values that get the y_i as close as possible to their expected values, in the sense of minimizing $Q(\boldsymbol{\beta}) = \sum_{i=1}^n (y_i - E_{\boldsymbol{\beta}}(y_i))^2$ over all $\boldsymbol{\beta}$ values. Following this recipe, obtain formulas for the least squares estimates of β_0 and β_1. Don't bother with second derivative tests. There is a better way to verify that you have found the minimum; we will cover it later.

(b) Suppose the ϵ_i are normally distributed. Using results from earlier in this assignment, what is the distribution of y_i?

(c) Starting from $\widehat{\beta}_1 = \frac{\sum_{i=1}^n (x_i - \overline{x})(y_i - \overline{y})}{\sum_{i=1}^n (x_i - \overline{x})^2}$, show $\widehat{\beta}_1 = \frac{\sum_{i=1}^n (x_i - \overline{x}) y_i}{\sum_{i=1}^n (x_i - \overline{x})^2}$.

(d) Using the preceding result,

i. What is the distribution of $\widehat{\beta}_1$ if the ϵ_i are normal?
ii. What is $Cov(\overline{y}, \widehat{\beta}_1)$?
iii. What is the distribution of $\widehat{\beta}_0 = \overline{y} - \widehat{\beta}_1 \overline{x}$ if the ϵ_i are normal?
iv. What is $Cov(\widehat{\beta}_0, \widehat{\beta}_1)$?

(e) Calculate $\widehat{\beta}_0$ and $\widehat{\beta}_1$ for the following data set. Your answers are numbers. Use R. You might be asked to use R on the quiz.

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x  0.0  1.3  3.2 -2.5 -4.6 -1.6  4.5  3.8
y -0.8 -1.3  7.4 -5.2 -6.5 -4.9  9.9  7.2
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