Assignment 1

(1)

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> x = -4:4; px = abs(x)/20; y = x^2-1
> cbind(x,x^2,px,y)
      x      px  y
 [1,] -4 16 0.20 15
 [2,] -3  9 0.15  8
 [3,] -2  4 0.10  3
 [4,] -1  1 0.05  0
 [5,]  0  0 0.00 -1
 [6,]  1  1 0.05  0
 [7,]  2  4 0.10  3
 [8,]  3  9 0.15  8
 [9,]  4 16 0.20 15
> Ex = sum(x*px); Ex # You know it's zero by symmetry
[1] 0
> Varx = sum(x^2*px); Varx
[1] 10
> # Use formula for E(g(x)) to get E(y) and Var(y)
> Ey = sum(y*px); Ey
[1] 9
> Vary = sum( (y-Ey)^2 * px ); Vary
[1] 30
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(f) $P(y=0) = P(x=1) + P(x=-1) = 1/20 + 1/20 = 1/10$

$P(y=3) = P(x=2) + P(x=-2) = 2/20 + 2/20 = 2/10$

$P(y=8) = P(x=3) + P(x=-3) = 3/20 + 3/20 = 3/10$

$P(y=15) = P(x=4) + P(x=-4) = 4/20 + 4/20 = 4/10$

(g) $E(y) = \sum_y y\, P(y) = 0 \cdot \frac{1}{10} + 3 \cdot \frac{2}{10} + 8 \cdot \frac{3}{10} + 15 \cdot \frac{4}{10}$

$= (6 + 24 + 60)/10 = 90/10 = 9$

(h) $Var(y) = \sum_y (y - \mu_y)^2 P(y)$

$= (0-9)^2 \cdot \frac{1}{10} + (3-9)^2 \cdot \frac{2}{10} + (8-9)^2 \cdot \frac{3}{10} + (15-9)^2 \cdot \frac{4}{10}$

$= 81 \cdot \frac{1}{10} + 36 \cdot \frac{2}{10} + 1 \cdot \frac{3}{10} + 36 \cdot \frac{4}{10}$

$= (81 + 72 + 3 + 144)/10 = 300/10 = 30$

Note (g) & (h) agree with R calculations.

(2) (a) $E(x) = \sum_x x\,p(x) = a \cdot 1 = a$

$Var(x) = \sum_x (x-\mu_x)^2 p(x) = (a-a)^2 \cdot 1 = 0$

(b) $E(y) = \int_{-\infty}^{\infty} a\,f(x)\,dx = a \int_{-\infty}^{\infty} f(x)\,dx = a \cdot 1 = a$

$$Var(y) = E(y - E(y))^2 = E\{(g(x)-a)^2\} = E\{h(x)\}$$

$$= \int_{-\infty}^{\infty} (a-a)^2 f(x)\,dx = \int_{-\infty}^{\infty} 0 \cdot f(x)\,dx$$

$$= \int_{-\infty}^{\infty} 0\,dx = 0$$

(3) (a and b) Write the marginal probabilities on the margins.

	$x=1$	$x=2$	$x=3$	
$y=1$	3/12	1/12	3/12	7/12
$y=2$	1/12	3/12	1/12	5/12
	4/12	4/12	4/12	

(c) $E(X) = (1+2+3)\cdot\frac{4}{12} = \frac{24}{12} = 2$

(d)

x	$x-2$	$(x-2)^2$	$p(x)$
1	-1	1	1/3
2	0	0	1/3
3	1	1	1/3

$Var(X) = \sum_x (x-\mu)^2 p(x) = (1+0+1)\cdot\frac{1}{3} = \frac{2}{3}$

(e) $E(Y) = 1\cdot\frac{7}{12} + 2\cdot\frac{5}{12} = \frac{17}{12}$

(f) $Var(Y) = E(Y^2) - [E(Y)]^2$, $E(Y^2) = 1^2\cdot\frac{7}{12} + 2^2\frac{5}{12}$
$= \frac{7}{12} + \frac{20}{12} = 27/12$

$Var(Y) = \frac{27}{12} - \left(\frac{17}{12}\right)^2 = \frac{324-289}{144} = \frac{35}{144}$

(3g)

x	y	$x+y$	$P(x,y)$
1	1	2	3/12
1	2	3	1/12
2	1	3	1/12
2	2	4	3/12
3	1	4	3/12
3	2	5	1/12

z_1	2	3	4	5
$P(z_1)$	3/12	2/12	6/12	1/12

(h) $E(z_1) = 2 \cdot \frac{3}{12} + 3 \cdot \frac{2}{12} + 4 \cdot \frac{6}{12} + 5 \cdot \frac{1}{12}$

$= (6+6+24+5)/12 = 41/12$

(i) $E(X) + E(Y) = 2 + \frac{17}{12} = \frac{24+17}{12} = \frac{41}{12}$ Yes

(3j)

x	y	xy	$P(x,y)$
1	1	1	3/12
1	2	2	1/12
2	1	2	1/12
2	2	4	3/12
3	1	3	3/12
3	2	6	1/12

z_2	1	2	3	4	6
$P(z_2)$	3/12	2/12	3/12	3/12	1/12

(k) $E(Z_2) = 1 \cdot \frac{3}{12} + 2 \cdot \frac{2}{12} + 3 \cdot \frac{3}{12} + 4 \cdot \frac{3}{12} + 6 \cdot \frac{1}{12}$

$= (3+4+9+12+6)/12 = 34/12 = 17/6$

(l) $E(X)E(Y) = 2 \cdot \frac{17}{12} = \frac{17}{6}$ Yes

(m) $Cov(X,Y) = E(XY) - E(X)E(Y) = 0$

(n) No : $P(X=3, Y=2) = \frac{1}{12} = \frac{12}{144}$,

But $P(X=3)P(Y=2) = \frac{4}{12} \cdot \frac{5}{12} = \frac{20}{144}$

There can be zero covariance without independence

(4) $E(X_1 X_2) = \int\int x_1 x_2 \, f_{X_1,X_2}(x_1, x_2)\, dx_1\, dx_2$

$= \int\int x_1 x_2 \, f_{X_1}(x_1)\, f_{X_2}(x_2)\, dx_1\, dx_2$

This is where I use independence

$= \int x_2 f_{X_2}(x_2) \int x_1 f_{X_1}(x_1)\, dx_1\, dx_2$

$= \int x_2 f_{X_2}(x_2)\, E(X_1)\, dx_2 = E(X_1) \int x_2 f_{X_2}(x_2)\, dx_2$

$= E(X_1) E(X_2)$

Yes, same proof replacing integrals with sums

5 (a) $Var(Y) \overset{def}{=} E(Y-\mu_Y)^2 = E(Y^2 - 2\mu_Y Y + \mu_Y^2)$

$= E(Y^2) - 2\mu_Y E(Y) + E(\mu_Y^2)$

$= E(Y^2) - 2\mu_Y^2 + \mu_Y^2 = E(Y^2) - \mu_Y^2$

(b) $Cov(X,Y) \overset{def}{=} E\{(X-\mu_X)(Y-\mu_Y)\}$

$= E(XY - X\mu_Y - \mu_X Y + \mu_X \mu_Y)$

$= E(XY) - E(X)\mu_Y - \mu_X E(Y) + \mu_X \mu_Y$

$= E(XY) - \mu_X \mu_Y - \mu_X \mu_Y + \mu_X \mu_Y$

$= E(XY) - \mu_X \mu_Y = E(XY) - E(X)E(Y)$

(c) By Problem 4, if X & Y are independent then

$E(XY) = E(X)E(Y) \Rightarrow \underbrace{E(XY) - E(X)E(Y)}_{\overset{\shortparallel}{Cov(X,Y) \text{ by } (5b)}} = 0$

6 (a) $Var(aX) = E\{(aX - E(aX))^2\}$

$= E\{(aX - aE(X))^2\} = E\{a^2(X-\mu_X)^2\}$

$= a^2 E\{(X-\mu_X)^2\} = a^2 Var(X)$

(b) $Var(X+a) = E\{(X+a - E(X+a))^2\}$

$= E\{(X+\cancel{a} - E(X)-\cancel{a})^2\} = E\{(X-\mu_X)^2\} = Var(X)$

7 $\mathrm{Var}(X+Y) = E\{(X+Y-E(X+Y))^2\}$

$$= E\{(X+Y-E(X)-E(Y))^2\} = E\{(X-\mu_X+Y-\mu_Y)^2\}$$

$$= E\left[(X-\mu_X)^2 + 2(X-\mu_X)(Y-\mu_Y) + (Y-\mu_Y)^2\right]$$

$$= E(X-\mu_X)^2 + 2E\left[(X-\mu_X)(Y-\mu_Y)\right] + E(Y-\mu_Y)^2$$

$$= \mathrm{Var}(X) + 2\,\mathrm{Cov}(X,Y) + \mathrm{Var}(Y)$$

8 $\mathrm{Cov}(X+a,\, Y+b) = E\{(X+a-E(X+a))(Y+b-E(Y+b))\}$

$$= E\{(X+\cancel{a}-E(X)-\cancel{a})(Y+\cancel{b}-E(Y)-\cancel{b})\}$$

$$= E\{(X-\mu_X)(Y-\mu_Y)\} = \mathrm{Cov}(X,Y)$$

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$\boxed{9}$

(9) (a) $\mathrm{Cov}(aX, Y) = E\{(aX - E(aX))(Y-\mu_Y)\}$

$= E\{a(X-E(X))(Y-\mu_Y)\}$

$= aE\{(X-\mu_X)(Y-\mu_Y)\} = a\,\mathrm{Cov}(X,Y)$

(b) $$\mathrm{Corr}(aX, Y) = \frac{\mathrm{Cov}(aX, Y)}{\sqrt{\mathrm{Var}(aX)\,\mathrm{Var}(Y)}}$$

$$= \frac{a\,\mathrm{Cov}(X,Y)}{\sqrt{a^2\mathrm{Var}(X)\,\mathrm{Var}(Y)}} = \frac{a}{|a|}\,\frac{\mathrm{Cov}(X,Y)}{\sqrt{\mathrm{Var}(X)\,\mathrm{Var}(Y)}}$$

$= \mathrm{Sign}(a)\,\mathrm{Corr}(X,Y)$

(10) $\operatorname{Cov}(X_1+X_2, Y_1+Y_2) = E\{(X_1+X_2-E(X_1+X_2))$
$(Y_1+Y_2-E(Y_1+Y_2))\}$

$= E\{(X_1+X_2-\mu_1-\mu_2)(Y_1+Y_2-\mu_3-\mu_4)\}$

$= E\{(X_1-\mu_1+X_2-\mu_2)(Y_1-\mu_3+Y_2-\mu_4)\}$

$= E\{(X_1-\mu_1)(Y_1-\mu_3)+(X_1-\mu_1)(Y_2-\mu_4)$
$+(X_2-\mu_2)(Y_1-\mu_3)+(X_2-\mu_2)(Y_2-\mu_4)\}$

$= E[(X_1-\mu_1)(Y_1-\mu_3)]+E[(X_1-\mu_1)(Y_2-\mu_4)]+E[(X_2-\mu_2)(Y_1-\mu_3)]$
$+E[(X_2-\mu_2)(Y_2-\mu_4)]$

(11) (a) $\sum_{i=1}^{n}(y_i-\bar{y}) = \sum_{i=1}^{n} y_i - \sum_{i=1}^{n} \bar{y} = \sum_{i=1}^{n} y_i - n\bar{y}$

$= \sum_{i=1}^{n} y_i - \cancel{n}\frac{\sum_{i=1}^{n} y_i}{\cancel{n}} = 0$

(b) $\sum_{i=1}^{n}(y_i-\bar{y})^2 = \sum_{i=1}^{n}(y_i^2 - 2y_i\bar{y} + \bar{y}^2)$

$= \sum_{i=1}^{n} y_i^2 - 2\bar{y}\sum_{i=1}^{n} y_i + \sum_{i=1}^{n}\bar{y}^2$

$= \sum_{i=1}^{n} y_i^2 - 2\bar{y}n\bar{y} + n\bar{y}^2 = \sum_{i=1}^{n} y_i^2 - n\bar{y}^2$

(c) $\frac{dQ_m}{dm} = \frac{d}{dm}\sum_{i=1}^{n}(y_i-m)^2 = \sum_{i=1}^{n}\frac{d}{dm}(y_i-m)^2$

$= \sum_{i=1}^{n} 2(y_i-m)(-1) = -2\left(\sum_{i=1}^{n} y_i - \sum_{i=1}^{n} m\right)$

$= -2(n\bar{y} - nm) = -2n(\bar{y}-m) \overset{set}{=} 0$

$\Longrightarrow m = \bar{y}$.

2nd derivative test $\frac{d^2 m}{dm^2} = \frac{d}{dm}\left(-2n\bar{y} + 2nm\right)$

$= 2n > 0$ concave up minimum

(12) $\sum_{i=1}^{n}(x_i-\bar{x})(y_i-\bar{y}) = \sum_{i=1}^{n}(x_iy_i - x_i\bar{y} - \bar{x}y_i + \bar{x}\bar{y})$

$= \sum_{i=1}^{n} x_iy_i - \bar{y}\sum_{i=1}^{n} x_i - \bar{x}\sum_{i=1}^{n} y_i + n\bar{x}\bar{y}$

$= \sum_{i=1}^{n} x_iy_i - n\bar{x}\bar{y} - n\bar{x}\bar{y} + n\bar{x}\bar{y}$

$= \sum_{i=1}^{n} x_iy_i - n\bar{x}\bar{y}$

(13)

(a) $E\left(\sum_{i=1}^{n} Y_i\right) = \sum_{i=1}^{n} E(Y_i) = \sum_{i=1}^{n} \mu = n\mu$

(b) $Var\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} Var(X_i)$ using Q5c & Q7

$= \sum_{i=1}^{n} \sigma^2 = n\sigma^2$

(c) $Var(\bar{Y}) = Var\left(\frac{1}{n}\sum_{i=1}^{n} Y_i\right) = \frac{1}{n^2} Var\left(\sum_{i=1}^{n} Y_i\right)$

$= \frac{1}{n^2} n\sigma^2 = \frac{\sigma^2}{n}$

(d) $E(\bar{Y}) = E\left(\frac{1}{n}\sum_{i=1}^{n} Y_i\right) = \frac{1}{n}\sum_{i=1}^{n} E(Y_i) = \frac{1}{n}\sum_{i=1}^{n} \mu$

$= \frac{1}{n} n\mu = \mu$

(e) L is unbiased iff $\mu = E(L) = E\left(\sum_{i=1}^{n} a_i Y_i\right)$

$= \sum_{i=1}^{n} a_i E(Y_i) = \sum_{i=1}^{n} a_i \mu$. If $\mu \neq 0$, this is true iff $\sum_{i=1}^{n} a_i = 1$. If $\mu = 0$, $E(L) = \mu$ regardless of a_i. So $\sum_{i=1}^{n} a_i = 1$ makes $E(L) = \mu$ for all $\mu \in \mathbb{R}$. That is, L will be unbiased.

(f) Yes: $a_i = \frac{1}{n}$, for $i = 1, \ldots, n$

(g) $Var(L) = Var\left(\sum_{i=1}^{n} a_i Y_i\right) = \sum_{i=1}^{n} a_i^2 Var(Y_i) = \sum_{i=1}^{n} a_i^2 \sigma^2$

$= \sigma^2 \sum_{i=1}^{n} a_i^2$

(14)

(a) $E(y_i) = E(\beta_0 + \beta_1 x_i + \varepsilon_i) = \beta_0 + \beta_1 x_i + E(\varepsilon_i)$

$= \beta_0 + \beta_1 x_i$

(b) $Var(y_i) = Var(\beta_0 + \beta_1 x_i + \varepsilon_i) = Var(\varepsilon_i) = \sigma^2$

(c) $f_{Y_i}(y) = \frac{d}{dy} F_{Y_i}(y) = \frac{d}{dy} P(Y_i \le y)$

$= \frac{d}{dy} P(\beta_0 + \beta_1 x_i + \varepsilon_i \le y) = \frac{d}{dy} P(\varepsilon_i \le y - \beta_0 - \beta_1 x_i)$

$= \frac{d}{dy} F_\varepsilon(y - \beta_0 - \beta_1 x_i) = f_\varepsilon(y_i - \beta_0 - \beta_1 x_i)$

$= \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2\sigma^2}(y - \beta_0 - \beta_1 x_i)^2}$

$N(\beta_0 + \beta_1 x_i, \sigma^2)$

(d) $E(\hat{\beta}_1) = E\left(\frac{\sum_{i=1}^n x_i Y_i}{\sum_{i=1}^n x_i^2}\right) = \frac{1}{\sum_{i=1}^n x_i^2} \sum_{i=1}^n x_i E(Y_i)$

$= \frac{1}{\sum_{i=1}^n x_i^2} \sum_{i=1}^n x_i(\beta_0 + \beta_1 x_i) = \frac{1}{\sum_{i=1}^n x_i^2}\left(\beta_0 \sum_{i=1}^n x_i + \beta_1 \sum_{i=1}^n x_i^2\right)$

$= \beta_0 \frac{\sum_{i=1}^n x_i}{\sum_{i=1}^n x_i^2} + \beta_1$ Biased

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(a) No	(b) No	(c) Yes
(d) Yes	(e) No	(f) No
(g) Yes	(h) Yes	(i) No

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$$A'B = \begin{pmatrix} 2 & 1 & 0 \\ 5 & -4 & 3 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 2 & 3 \\ -1 & 3 \end{pmatrix} = \begin{pmatrix} 2+2+0 & 0+3+0 \\ 5-8-3 & 0-12+9 \end{pmatrix}$$

17 $$c'd = \begin{pmatrix} 2 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} = (2+2) = (4)$$

$$cd' = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 2 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 2\cdot 1 & 2\cdot 2 & 2\cdot -1 \\ 1\cdot 1 & 1\cdot 2 & 1\cdot -1 \\ 0\cdot 1 & 0\cdot 2 & 0\cdot -1 \end{pmatrix}$$

(18) a (19) c (20) b (21) b (22) c

(23) b (24) d (25) d

(26) A^{-1} does not exist.

(27) $(X'X)' = X'X'' = X'X$

(28)
$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \text{ but}$$
$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$$

(29) Only square matrices can have inverses.

(30) (a) $AB = \begin{pmatrix} 4 & 4 \\ 8 & 8 \end{pmatrix}$ and $AC = \begin{pmatrix} 4 & 4 \\ 8 & 8 \end{pmatrix}$

(b) Yes

(c) Just kidding. But some students would do this

$$\cancel{A}B = \cancel{A}C \Rightarrow B = C$$

and get a zero.