Assignment 12

① $f_{\varepsilon|\underset{\sim}{x}}(\varepsilon|\underset{\sim}{x}) = f_{\varepsilon}(\varepsilon)$ (normal)

$$\Longrightarrow \frac{f_{\varepsilon,\underset{\sim}{x}}(\varepsilon,\underset{\sim}{x})}{f_{\underset{\sim}{x}}(\underset{\sim}{x})} = f_{\varepsilon}(\varepsilon)$$

$\Longrightarrow f_{\varepsilon,\underset{\sim}{x}}(\varepsilon,\underset{\sim}{x}) = f_{\varepsilon}(\varepsilon)\, f_{\underset{\sim}{x}}(\underset{\sim}{x})$ independent, so

$\operatorname{cov}(\varepsilon, \underset{\sim}{x}) = \underset{\sim}{0}$.

② $E(\varepsilon_i) = E(E(\varepsilon_i | X_i)) = E(0) = 0$, so

$$\operatorname{Cov}(\varepsilon_i, X_i) = E(\varepsilon_i X_i) - \underbrace{E(\varepsilon_i)}_{0} E(X_i) = E(\varepsilon_i X_i)$$

$$= \int\int \varepsilon x \; f_{\varepsilon,x}(\varepsilon, x)\, d\varepsilon\, dx$$

$$= \int x \int \varepsilon \frac{f_{\varepsilon,x}(\varepsilon,x)}{f_x(x)}\, d\varepsilon\, f_x(x)\, dx$$

$$= \int x \int \varepsilon\, f_{\varepsilon|x}(\varepsilon|x)\, d\varepsilon\, f_x(x)\, dx$$

$$= \int x\, E(\varepsilon|x)\, f_x(x)\, dx = \int x * 0 \cdot f_x(x)\, dx$$

$$= 0$$

OR, $E(\varepsilon X) = E\{E(\varepsilon X | X)\}$

$= E\{X E(\varepsilon|X)\} = E\{X * 0\} = 0$

③ (a) $E(\hat{\beta} \mid \mathcal{X} = x) = E\left((\mathcal{X}'\mathcal{X})^{-1}\mathcal{X}'y \mid \mathcal{X} = x\right)$

$= E\left((x'x)^{-1}x'y \mid \mathcal{X} = x\right) = (x'x)^{-1}x'E(y \mid \mathcal{X} = x)$

$= (x'x)^{-1}x'x\beta = \beta$

(b) $E(\hat{\beta}) = E\{E(\hat{\beta} \mid \mathcal{X})\} = E\{\beta\} = \beta$

(c) $P(F > f_c) \overset{H_0}{=} \underset{x}{\sum\cdots\sum} P(F > f_c \mid \mathcal{X} = x)\, P(\mathcal{X} = x)$

$= \underset{x}{\sum\cdots\sum} \alpha\, P(\mathcal{X} = x) = \alpha \underset{x}{\sum\cdots\sum} P(\mathcal{X} = x)$

$= \alpha * 1 = \alpha$

(4) $y_i = \alpha + \beta' x_i + \varepsilon_i$

(a) $\text{cov}(x_i, y_i) = \text{cov}(x_i, \alpha + \beta' x_i + \varepsilon_i)$

$= \text{cov}(x_i, \beta' x_i) + \text{cov}(x_i, \varepsilon_i) = \text{cov}(x_i, x_i)\beta'' + 0$

$= \Sigma_x \beta$

(b) $\Sigma_x \beta = \Sigma_{xy} \Rightarrow \beta = \Sigma_x^{-1} \Sigma_{xy}$

(c) How about $\hat{\beta} = \hat{\Sigma}_x^{-1} \hat{\Sigma}_{xy}$

(5) (a) $\text{cov}(x_{i1}, \varepsilon_i^*) = \text{cov}(x_{i1}, \beta_2 x_{i2} - \beta_2 \mu_2 + \varepsilon_i)$

$= \text{cov}(x_{i1}, \beta_2 x_{i2} + \varepsilon_i) = \beta_2 \text{cov}(x_{i1}, x_{i2}) + \text{cov}(x_{i1}, \varepsilon_i)$

$= \beta_2 \sigma_{12} + 0 = \beta_2 \sigma_{12}$

(b) $\text{cov}(x_{i1}, y_i) = \text{cov}(x_{i1}, \beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + \varepsilon_i)$

$= \text{cov}(x_{i1}, \beta_1 x_{i1} + \beta_2 x_{i2} + \varepsilon_i)$

$= \beta_1 \text{cov}(x_{i1}, x_{i1}) + \beta_2 \text{cov}(x_{i1}, x_{i2}) + \text{cov}(x_{i1}, \varepsilon_i)$

$= \beta_1 \sigma_{11} + \beta_2 \sigma_{12}$

(c) Nonzero covariance is definitely possible when $\beta_1 = 0$, provided β_2 & σ_{12} are both non-zero.

(5c)
$$\hat{\beta}_1 = \frac{\sum_{i=1}^{n} (x_{i1} - \bar{x}_1)(y_i - \bar{y})}{\sum_{i=1}^{n} (x_{i1} - \bar{x}_1)^2}$$

$$= \frac{\frac{1}{n-1} \sum_{i=1}^{n} (x_{i1} - \bar{x}_1)(y_i - \bar{y})}{\frac{1}{n-1} \sum_{i=1}^{n} (x_{i1} - \bar{x}_1)^2}$$

$$= \frac{\hat{\sigma}_{x_1 y}}{\hat{\sigma}_{x_1}^2} \longrightarrow \frac{\beta_1 \phi_{11} + \beta_2 \phi_{12}}{\phi_{11}}$$

By continuous mappings (And Law of Large numbers)

$$= \beta_1 + \beta_2 \frac{\phi_{12}}{\phi_{11}} \neq \beta_1 \quad \text{in general,}$$

So <u>no</u> we do not have $\hat{\beta}_1 \longrightarrow \beta_1$

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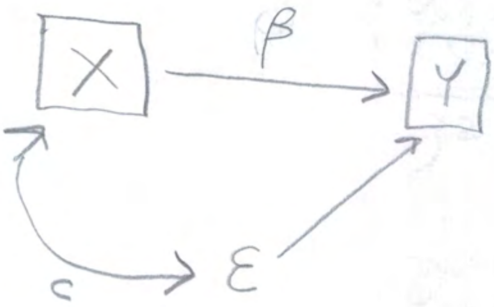
(a)



(b)

$$\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \beta & 1 \end{pmatrix} \begin{pmatrix} X \\ \varepsilon \end{pmatrix} = A \begin{pmatrix} X \\ \varepsilon \end{pmatrix}$$

$$\sim N(A\mu, A\Sigma A'),$$

$$A\mu = \begin{pmatrix} 1 & 0 \\ \beta & 1 \end{pmatrix} \begin{pmatrix} \mu_x \\ 0 \end{pmatrix} = \begin{pmatrix} \mu_x \\ \beta \mu_x \end{pmatrix}$$

And $A \Sigma A' =$

$$\begin{pmatrix} 1 & 0 \\ \beta & 1 \end{pmatrix} \begin{pmatrix} \sigma_x^2 & c \\ c & \sigma_\varepsilon^2 \end{pmatrix} \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \sigma_x^2 & c \\ \beta\sigma_x^2 + c & \beta c + \sigma_\varepsilon^2 \end{pmatrix} \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \sigma_x^2 & \beta\sigma_x^2 + c \\ \beta\sigma_x^2 + c & \beta^2\sigma_x^2 + \beta c \\ & + \beta c + \sigma_\varepsilon^2 \end{pmatrix}$$

So

$$\begin{pmatrix} X \\ Y \end{pmatrix} \sim N\left[\begin{pmatrix} \mu_x \\ \beta\mu_x \end{pmatrix}, \begin{pmatrix} \sigma_x^2 & \beta\sigma_x^2 + c \\ \beta\sigma_x^2 + c & \beta^2\sigma_x^2 + 2\beta c + \sigma_\varepsilon^2 \end{pmatrix} \right.$$

(6c) $\theta = (\beta, \mu_x, \sigma_x^2, c, \sigma_\varepsilon^2)$

(d) Set

	β	μ_x	σ_x^2	c	σ_ε^2
θ_1	0	0	4	4	5
θ_2	1	0	4	0	1

Both yield $E\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and

$$cov\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} \sigma_x^2 & \beta\sigma_x^2 + c \\ \beta\sigma_x^2 + c & \beta^2\sigma_x^2 + 2\beta c + \sigma_\varepsilon^2 \end{pmatrix} = \begin{pmatrix} 4 & 4 \\ 4 & 5 \end{pmatrix}$$

And so the same bivariate normal distribution of X & Y. You can __never__ tell if $\beta = 0$ or not.

How did I come up with this example of $\theta_1 \neq \theta_2$? Looking at the path diagram in 6(a), it seems X can be related to Y directly through β or indirectly through c. This is bad news for estimating and testing hypotheses about β. Anyway, one of my parameter vectors will have $\beta = 0$ and $c \neq 0$, while the other will have $c = 0$ & $\beta \neq 0$. μ_x does not matter, so for simplicity make it zero in both vectors. Looking at $cov\begin{pmatrix} X \\ Y \end{pmatrix}$ see σ_x^2 must be the same in both vectors, so set $\sigma_x^2 = 4$, because why not? Also from $cov\begin{pmatrix} X \\ Y \end{pmatrix}$ want $\beta\sigma_x^2 = c$, so when $\beta \neq 0$, let $\beta = 1$ for simplicity. That yields $c = 4$. Finally, $var(Y) = \beta^2\sigma_x^2 + 0 + \sigma_\varepsilon^2 = 4\beta^2 + \sigma_\varepsilon^2$. When $\beta = 1$, let $\sigma_\varepsilon^2 = 1$, for a variance of 5. When $\beta = 0$, this makes $\sigma_\varepsilon^2 = 5$. Done.

Well, almost done. While β could be anything, c is constrained by $1 \geq |corr(X, \varepsilon)| = \frac{|cov(X,\varepsilon)|}{\sqrt{\sigma_x^2 \sigma_\varepsilon^2}} = \frac{|c|}{\sqrt{\sigma_x^2\sigma_\varepsilon^2}} \Leftrightarrow c^2 \leq \sigma_x^2 \sigma_\varepsilon^2$

When $c = 0$, there is no problem. For $c = 4$, $16 \leq 4 \cdot 5 = 20$, __OKAY__.

7 (a) See slide 21 of lecture slide set 25.

(b) $$\hat{\beta}_1 = \frac{\hat{\sigma}_{wx}}{\hat{\sigma}_w^2} = \frac{\sum_{i=1}^{n}(w_i - \bar{w})(x_i - \bar{x})}{\sum_{i=1}^{n}(w_i - \bar{w})^2}$$

$$\hat{v}_{12}/\hat{v}_{11}$$

The least squares estimator (check formula sheet)

(c) Looking at the covariance matrix,

$$\beta_2 = \frac{v_{13}}{v_{12}}, \quad v_{23} = \beta_2 v_{22} + c$$

$$\Rightarrow c = v_{23} - \frac{v_{13}}{v_{12}} v_{22}, \text{ so } \hat{\beta}_2 = \hat{v}_{23} - \frac{\hat{v}_{13}\hat{v}_{22}}{\hat{v}_{12}}$$

(d) Definitely, provided $\beta_1 \neq 0$. By continuous mapping $\hat{c} = \hat{v}_{23} - \dfrac{\hat{v}_{13}\hat{v}_{22}}{\hat{v}_{12}}$

$$\rightarrow v_{23} - \frac{v_{13} v_{22}}{v_{12}} = \beta_2(\beta_1^2\sigma_w^2 + \sigma_1^2) + c - \frac{\cancel{\beta_1}\beta_2\cancel{\sigma_w^2}(\beta_1^2\sigma_w^2 + \sigma_1^2)}{\cancel{\beta_1}\cancel{\sigma_w^2}}$$

$$= c$$