Assignment 11

(1) (a) If $\text{cov}(\varepsilon) = \sigma^2 V$, $E(y)$ is still $X\beta$, so

$$E(\hat{\beta}) = E\left((X'X)^{-1}X'y\right) = (X'X)^{-1}X'E(y)$$
$$= (X'X)^{-1}X'X\beta = \beta, \text{ so yes.}$$

(b) $$\text{cov}(\hat{\beta}) = \text{cov}\left((X'X)^{-1}X'y\right) = (X'X)^{-1}X'\,\text{cov}(y)\left((X'X)^{-1}X'\right)'$$
$$= (X'X)^{-1}X'\sigma^2 VX(X'X)^{-1'}$$
$$= \sigma^2(X'X)^{-1}X'VX(X'X)^{-1}$$

(c) $$y = X\beta + \varepsilon \Rightarrow V^{-\frac{1}{2}}y = V^{-\frac{1}{2}}X\beta + V^{-\frac{1}{2}}\varepsilon$$
$$\Rightarrow y^* = X^*\beta + \varepsilon^*$$

$$\text{cov}(\varepsilon^*) = \text{cov}(V^{-\frac{1}{2}}\varepsilon) = V^{-\frac{1}{2}}\text{cov}(\varepsilon)V^{-\frac{1}{2}'}$$
$$= V^{-1/2}\sigma^2 V V^{-\frac{1}{2}} = \sigma^2 I_n$$

(d) $$\hat{\beta}_{gls} = (X^{*'}X^*)^{-1}X^{*'}y^* = \left((V^{-\frac{1}{2}}X)'V^{-\frac{1}{2}}X\right)^{-1}(V^{-\frac{1}{2}}X)'V^{-\frac{1}{2}}y$$
$$= (X'V^{-\frac{1}{2}'}V^{-\frac{1}{2}}X)^{-1}X'V^{-\frac{1}{2}'}V^{-\frac{1}{2}}y$$
$$= (X'V^{-\frac{1}{2}}V^{-\frac{1}{2}}X)^{-1}X'V^{-\frac{1}{2}}V^{-\frac{1}{2}}y$$
$$= (X'V^{-1}X)^{-1}X'V^{-1}y$$

(1e) Because $\hat{\beta}_{gls} = Ay$ & $y \sim N_n(X\beta, \sigma^2 V)$, $\hat{\beta}_{gls}$ is multivariate normal.

$$E(\hat{\beta}_{gls}) = E\{(X'V^{-1}X)^{-1}X'V^{-1}y\}$$

$$= (X'V^{-1}X)^{-1}X'V^{-1}E(y) = \underbrace{(X'V^{-1}X)^{-1}X'V^{-1}X}\beta$$

$$= \beta \quad \text{and}$$

$$\text{cov}(\hat{\beta}_{gls}) = \text{cov}\{(X'V^{-1}X)^{-1}X'V^{-1}y\}$$

$$= (X'V^{-1}X)^{-1}X'V^{-1}\text{cov}\{y\}((X'V^{-1}X)^{-1}X'V^{-1})'$$

$$= (X'V^{-1}X)^{-1}X'V^{-1}\sigma^2 V\, V^{-1'}X\,(X'V^{-1}X)^{-1'}$$

$$= \sigma^2(X'V^{-1}X)^{-1}X'V^{-1}V\,V^{'-1}X(X'V^{-1}X)^{'-1}$$

$$= \sigma^2(X'V^{-1}X)^{-1}X'V^{-1}VV^{-1}X(X'V^{-1'}X)^{-1}$$

$$= \sigma^2\underbrace{(X'V^{-1}X)^{-1}X'V^{-1}X}_{I}\,(X'V^{'-1}X)^{-1}$$

$$= \sigma^2(X'V^{-1}X)^{-1}, \text{ so}$$

$$\hat{\beta}_{gls} \sim N(\beta, \sigma^2(X'V^{-1}X)^{-1})$$

(1f) Yes, Because $E(y) = X\beta$ and the calculation of $E(\hat{\beta}_{gls})$ is the same as in Problem 1e.

(g) i) $H^* = X^*(X^{*\prime}X^*)^{-1}X^{*\prime}$

$$= V^{-\frac{1}{2}}X\left((V^{-\frac{1}{2}}X)'V^{-\frac{1}{2}}X\right)^{-1}(V^{-\frac{1}{2}}X)'$$

$$= V^{-\frac{1}{2}}X(X'V^{-1}X)^{-1}X'V^{-\frac{1}{2}}$$

symmetric? $\left(V^{-\frac{1}{2}}X(X'V^{-1}X)^{-1}X'V^{-\frac{1}{2}}\right)'$

$$= V^{-\frac{1}{2}\prime}X(X'V^{-1}X)^{-1\prime}X'V^{-\frac{1}{2}\prime}$$

$$= V^{-\frac{1}{2}}X(X'V^{-1}X)^{-1}X'V^{-\frac{1}{2}} = H^*, \ \underline{\text{Yes}}$$

Idempotent? H^*H^*

$$= V^{-\frac{1}{2}}X(X'V^{-1}X)^{-1}X'\underbrace{V^{-\frac{1}{2}}\,V^{-\frac{1}{2}}}X(X'V^{-1}X)^{-1}X'V^{-\frac{1}{2}}$$

$$= V^{-\frac{1}{2}}X(X'V^{-\frac{1}{2}}X)^{-1}\underbrace{X'V^{-1}X\,(X'V^{-1}X)^{-1}}X'V^{-\frac{1}{2}}$$

$$= V^{-\frac{1}{2}}X(X'V^{-1}X)^{-1}X'V^{-\frac{1}{2}} = H^*, \ \underline{\text{Yes}}$$

(1g ii) $\hat{y}^* = X^* \hat{\beta}_{gls} = V^{-\frac{1}{2}} X (X' V^{-1} X)^{-1} X' V^{-1} y$

$= H^* y^*$ OKAY

(iii) $\hat{\varepsilon}^* = y^* - \hat{y}^* = V^{-\frac{1}{2}} y - V^{-\frac{1}{2}} X (X' V^{-1} X)^{-1} V^{-1} y$

$= \left(I - V^{-\frac{1}{2}} X (X' V^{-1} X)^{-1} V^{-\frac{1}{2}}\right) V^{-\frac{1}{2}} y$

$= (I - H^*) y^*$ OKAY

(iv) writing $\hat{\varepsilon}^* = y^* - X^* \hat{\beta}_{gls} = V^{-\frac{1}{2}} (y - X \hat{\beta}_{gls})$,

$SSE^* = \hat{\varepsilon}^{*\prime} \hat{\varepsilon} = \left(V^{-\frac{1}{2}} (y - X \hat{\beta}_{gls})\right)' V^{-\frac{1}{2}} (y - X \hat{\beta}_{gls})$

$= (y - X \hat{\beta}_{gls})' V^{-\frac{1}{2}\prime} V^{-\frac{1}{2}} (y - X \hat{\beta}_{gls})$

$= (y - X \hat{\beta}_{gls})' V^{-1} (y - X \hat{\beta}_{gls})$ as required

(1gv)

$$F^* = \frac{(C\hat{\beta}_{gls} - t)'(C(X^{*\prime}X^*)^{-1}C')^{-1}(C\hat{\beta}_{gls} - t)}{q\,MSE^*}$$

$$= \frac{(C\hat{\beta}_{gls} - t)'(C(X'V^{-1}X)^{-1}C')^{-1}(C\hat{\beta}_{gls} - t)}{\left(\frac{q}{n-k-1}\right)(y - X\hat{\beta}_{gls})'V^{-1}(y - X\hat{\beta}_{gls})}$$

2 (a) $y_i \sim N(\beta_0 + \beta_1 x_{i1} + \cdots + \beta_k x_{ik}, \sigma^2 v_i)$

(b) i) $x^*_{i,0} = \frac{1}{\sqrt{v_i}}$

ii) $E(\varepsilon_i^*) = E\left(\frac{1}{\sqrt{v_i}} \varepsilon_i\right) = 0$ and

$$Var(\varepsilon_i^*) = Var\left(\frac{1}{\sqrt{v_i}} \varepsilon_i\right) = \frac{1}{v_i} Var(\varepsilon_i)$$

$$= \frac{1}{v_i} \sigma^2 v_i = \sigma^2$$

So $\varepsilon_i^* \sim N(0, \sigma^2)$, still independent, so

$\varepsilon_1^*, \ldots, \varepsilon_n^* \stackrel{iid}{\sim} N(0, \sigma^2)$

Because functions of independent random variables are independent.

(2c) Least squares estimates for starred model are obtained by minimizing

$$Q(\hat{\beta}) = \sum_{i=1}^{n} \left(y_i^* - \hat{\beta}_0 x_{i0}^* - \hat{\beta}_1 x_{i1}^* - \cdots - \hat{\beta}_k x_{ik}^* \right)^2$$

$$= \sum_{i=1}^{n} \left(\frac{1}{\sqrt{v_i}} y_i - \hat{\beta}_0 \frac{1}{\sqrt{v_i}} - \hat{\beta}_1 \frac{1}{\sqrt{v_i}} x_{i1} - \cdots - \hat{\beta}_k \frac{1}{\sqrt{v_i}} x_{ik} \right)^2$$

$$= \sum_{i=1}^{n} \left(\frac{1}{\sqrt{v_i}} \left(y_i - \hat{\beta}_0 - \hat{\beta}_1 x_{i1} - \cdots - \hat{\beta}_k x_{ik} \right) \right)^2$$

$$= \sum_{i=1}^{n} \frac{1}{v_i} \left(y_i - \hat{\beta}_0 - \hat{\beta}_1 x_{i1} - \cdots - \hat{\beta}_k x_{ik} \right)^2$$

$$= \sum_{i=1}^{n} \frac{1}{v_i} \hat{\varepsilon}_i^2 = \sum_{i=1} w_i \hat{\varepsilon}_i^2 \text{, where}$$

$$w_i = \frac{1}{v_i} \quad \text{for } i = 1, \ldots, n$$

(3) $\bar{y}_j = \mu + \varepsilon_j$

$$\bar{y} = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} \mu + \begin{pmatrix} \varepsilon_1 \\ \vdots \\ \varepsilon_m \end{pmatrix}$$

$$V = \begin{pmatrix} \frac{1}{n_1} & & 0 \\ & \frac{1}{n_2} & \\ 0 & & \ddots \ \frac{1}{n_m} \end{pmatrix} \ \& \ V^{-1} = \begin{pmatrix} n_1 & & 0 \\ & n_2 & \\ 0 & & \ddots \ n_m \end{pmatrix}$$

$$\hat{\beta}_{gls} = (X'V^{-1}X)^{-1} X'V^{-1}\bar{y}$$

$$= \left((1 \cdots 1) V^{-1} \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \right)^{-1} (1 \cdots 1) V^{-1} \begin{pmatrix} \bar{y}_1 \\ \vdots \\ \bar{y}_m \end{pmatrix}$$

$$= \left(\sum_{j=1}^{m} n_j \right)^{-1} (n_1 \ n_2 \cdots n_m) \begin{pmatrix} \bar{y}_1 \\ \bar{y}_2 \\ \vdots \\ \bar{y}_m \end{pmatrix}$$

$$= \frac{\sum_{j=1}^{m} n_j \bar{y}_j}{\sum_{j=1}^{m} n_j}$$ checks.

4 (a) $y_i = \beta x_i + \varepsilon_i$

$$\Longrightarrow \frac{1}{x_i} y_i = \beta \frac{1}{x_i} x_i + \frac{1}{x_i} \varepsilon_i$$

$$\Longrightarrow y_i^* = \beta + \varepsilon_i^* \, , \quad \mathrm{var}(\varepsilon_i^*) = \sigma^2$$

So $y_1^*, \ldots, y_n^* \overset{iid}{\sim} N(\beta, \sigma^2)$ and the least squares estimate of β is $\bar{y}^* = \frac{1}{n} \sum_{i=1}^{n} \frac{y_i}{x_i}$

Using calculus, minimize the quantity of Question 2c

$$Q(\hat{\beta}) = \sum_{i=1}^{n} w_i \hat{\varepsilon}_i^2 = \sum_{i=1}^{n} \frac{1}{x_i^2} (y_i - \hat{\beta} x_i)^2$$

$$\frac{\partial Q}{\partial \hat{\beta}} = \sum_{i=1}^{n} \frac{1}{x_i^2} 2 (y_i - \hat{\beta} x_i)(-x_i)$$

$$= -2 \sum_{i=1}^{n} \frac{1}{x_i} (y_i - \hat{\beta} x_i)$$

$$= -2 \left(\sum_{i=1}^{n} \frac{y_i}{x_i} - n\hat{\beta} \right) \overset{\text{set}}{=} 0$$

$$\Longrightarrow \hat{\beta} = \frac{1}{n} \sum_{i=1}^{n} \frac{y_i}{x_i}$$

2nd derivative test

$$\frac{\partial^2 Q}{\partial \hat{\beta}^2} = -2(0 - n) = 2n > 0$$

Concave up, <u>MINIMUM</u>

(4b) Have $X = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$, $V = \begin{pmatrix} x_1^2 & & & 0 \\ & x_2^2 & & \\ & & \ddots & \\ 0 & & & x_n^2 \end{pmatrix}$

$$\hat{\beta}_{gls} = (X'V^{-1}X)^{-1} X'V^{-1}y$$

$$= \left((x_1 \cdots x_n) \begin{pmatrix} 1/x_1^2 & & & 0 \\ & 1/x_2^2 & & \\ & & \ddots & \\ 0 & & & 1/x_n^2 \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \right)^{-1}$$

$$(x_1 \cdots x_n) \begin{pmatrix} 1/x_1^2 & & & 0 \\ & 1/x_2^2 & & \\ & & \ddots & \\ 0 & & & 1/x_n^2 \end{pmatrix} \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$$

$$= \left(\left(\tfrac{1}{x_1}\ \tfrac{1}{x_2} \cdots \tfrac{1}{x_n}\right) \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \right)^{-1} \left(\tfrac{1}{x_1}\ \tfrac{1}{x_2} \cdots \tfrac{1}{x_n}\right) \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}$$

$$= n^{-1} \left(\frac{y_1}{x_1} + \frac{y_2}{x_2} + \cdots + \frac{y_n}{x_n} \right)$$

$$= \frac{1}{n} \sum_{i=1}^{n} \frac{y_i}{x_i}$$ Checks

(5) (a) $\hat{\beta} = 0.2014$. Not quite: $P = 0.0524$

(b) $\hat{\beta}_{WLS} = 0.2304$

Yes, $P = 0.0263$

(c) $\text{mean}(y/x) = 0.2304109$

6 (a) $y_i = \beta_0 + \beta_1 x_i - \beta_1 \bar{x} + \beta_1 \bar{x} + \varepsilon_i$

$$= (\beta_0 + \beta_1 \bar{x}) + \beta_1 (x_i - \bar{x}) + \varepsilon_i$$

$$= \alpha_0 + \alpha_1 (x_i - \bar{x}) + \varepsilon_i$$

so $\alpha_0 = \beta_0 + \beta_1 \bar{x}$, $\alpha_1 = \beta_1$

(b) For the uncentered model, $X = \begin{pmatrix} 1 & x_1 \\ 1 & x_2 \\ \vdots & \vdots \\ 1 & x_n \end{pmatrix}$

For the centered model,

$$W = \begin{pmatrix} 1 & x_1 - \bar{x} \\ 1 & x_2 - \bar{x} \\ \vdots & \vdots \\ 1 & x_n - \bar{x} \end{pmatrix}$$

(c)

$$\begin{pmatrix} 1 & x_1 \\ 1 & x_2 \\ \vdots & \vdots \\ 1 & x_n \end{pmatrix} \begin{pmatrix} 1 & -\bar{x} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & x_1 - \bar{x} \\ 1 & x_2 - \bar{x} \\ \vdots & \vdots \\ 1 & x_n - \bar{x} \end{pmatrix}$$

(6d) Verifying that $\begin{pmatrix} 1 & \bar{x} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & -\bar{x} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$,

$$A^{-1} \quad A \quad = \quad I$$

$$A^{-1}\beta = \begin{pmatrix} 1 & \bar{x} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \beta_0 \\ \beta_1 \end{pmatrix} = \begin{pmatrix} \beta_0 + \beta_1 \bar{x} \\ \beta_1 \end{pmatrix} = \begin{pmatrix} \alpha_0 \\ \alpha_1 \end{pmatrix}$$

(7) $Q(\underset{\sim}{\beta}) = \sum_{i=1}^{n} \left(y_i - \beta_0 - \beta_1(x_{i1} - \bar{x}_1) - \cdots - \beta_k(x_{ik} - \bar{x}_k)\right)^2$

$$\frac{\partial Q}{\partial \beta_0} = \sum_{i=1}^{n} 2\left(y_i - \beta_0 - \beta_1(x_{i1} - \bar{x}_1) - \cdots - \beta_k(x_{ik} - \bar{x}_k)\right)(-1)$$

$$= -2\left(\sum_{i=1}^{n} y_i - \sum_{i=1}^{n} \beta_0 - \beta_1 \underbrace{\sum_{i=1}^{n}(x_{i1} - \bar{x}_1)}_{=0} - \cdots - \beta_k \underbrace{\sum_{i=1}^{n}(x_{ik} - \bar{x}_k)}_{=0}\right)$$

$$= -2\left(\sum_{i=1}^{n} y_i - n\beta_0 - 0 - 0 \cdots - 0\right)$$

$$= -2\left(\sum_{i=1}^{n} y_i - n\beta_0\right) \stackrel{\text{set}}{=} 0 \implies \beta_0 = \frac{\sum_{i=1}^{n} y_i}{n} = \bar{y}$$

2nd Derivative test

$$\frac{\partial^2 Q}{\partial \beta_0^2} = -2(0 - n) = 2n > 0 \; \checkmark$$

concave up, minimum, so

$$\hat{\beta}_0 = \bar{y}$$

8 $\boxed{14}$

(a) $\hat{\alpha} = (W'W)^{-1}W'y$

$= ((XA)'XA)^{-1}(XA)'y = (A'X'XA)^{-1}A'X'y$

$= A^{-1}(X'X)^{-1}\underbrace{(A')^{-1}A'}_{I}X'y = A^{-1}(X'X)^{-1}X'y$

$= A^{-1}\hat{\beta}$

(b) $\hat{y} = W\hat{\alpha} = XAA^{-1}\hat{\beta} = X\hat{\beta}$

Same $\hat{y}$

(c) $H_0: C\beta = t \iff CAA^{-1}\beta = t$

$\iff CA\alpha = t \iff C_2\alpha = t, \quad C_2 = CA$

(d) For $H_0: C_2\alpha = t$,

$F^* = (C_2\hat{\alpha} - t)'(C_2(W'W)^{-1}C_2')^{-1}(C_2\hat{\alpha} - t)/(q\,MSE)$

$= (C\underbrace{AA^{-1}}_{I}\hat{\beta} - t)(CA((XA)'XA)^{-1}(CA)')^{-1}(CAA^{-1}\hat{\beta} - t)/(q\,MSE)$

$= (C\hat{\beta} - t)'(CA(A'X'XA)^{-1}A'C')^{-1}(C\hat{\beta} - t)/(q\,MSE)$

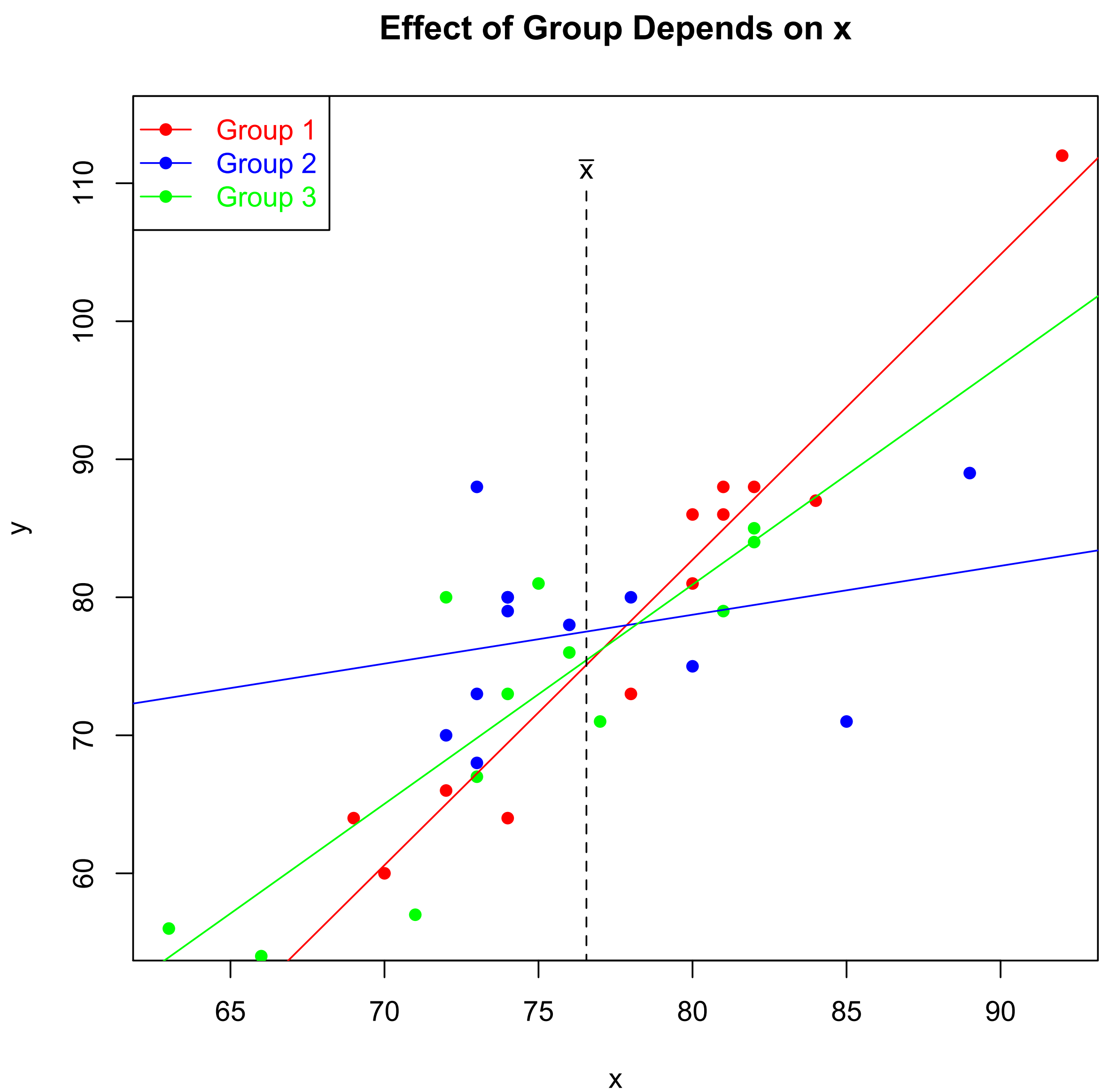
$= (C\hat{\beta} - t)'(C\underbrace{AA^{-1}}_{I}(X'X)^{-1}\underbrace{(A')^{-1}A'}_{I}C')^{-1}(C\hat{\beta} - t)/(q\,MSE)$

$= F^*$ for $H_0: C\beta = t$. Note MSE is the same because $\hat{y}$ is the same.

9. For this question, I needed to look at the following quite a few times.

$$y = \beta_0 + \beta_1 x + \beta_2 d_1 + \beta_3 d_2 + \beta_4 d_1 x + \beta_5 d_2 x + \epsilon$$

Package	d_1	d_2	$E(y\|\mathbf{x})$
1	1	0	$(\beta_0 + \beta_2) + (\beta_1 + \beta_4)x$
2	0	1	$(\beta_0 + \beta_3) + (\beta_1 + \beta_5)x$
3	0	0	$\beta_0 + \beta_1 x$



(9a) Looking at the table and setting the three expected values equal at $x = \bar{x}$, get

$H_0: \beta_2 + \beta_4 \bar{x} = \beta_3 + \beta_5 \bar{x} = 0$

Using ftest(), get

$F = 0.616$, $p = 0.5468$

There is no evidence that type of software package affects sales this quarter, for sales reps with average performance last quarter.

(b) $F = 0.616$ again, and it's a lot easier.

(c) Get $F = 5.889$, $p = 0.00696$

For sales reps who sold 85 units last quarter, type of software package has an effect on sales this quarter.

(d) For package 1 vs 3, I get $t = 1.177$, $p = 0.2485$

For package 2 vs 3, I get $t = -1.802$, $p = 0.0816$

For package 1 vs 2, $F = 11.778$, $p = 0.0018$

With the Bonferroni correction, $p = 0.0018 * 3$
≈ 0.005, so

Conclude that for representatives with sales last quarter of 85 units, use of software package 1 would be expected to yield higher sales this quarter than software package 2.

Get the directional conclusion from looking at the plot, or looking at the table and noting that $\beta_2 > 0$ while $\beta_3 < 0$.